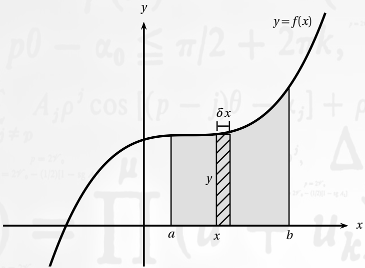
Area under a curve

Calculus 8

INU0115/515 (Maths 2)

Dr Adrian Jannetta MIMA CMath FRAS

The area beneath a curve



Consider the curve defined by $y = f(x)$.

The area enclosed by the curve, the x-axis and the lines $x = a$ and $x = b$ can be estimated by dividing it into rectangular strips.

The area of each strip δA is given by

$$\delta A = y\delta x = f(x)\delta x$$

The total area beneath the curve is approximately the sum of all such strips:

$$\text{Area} \approx \sum_{x=a}^{x=b} f(x)\delta x \qquad (1)$$

The estimate of area is improved by using narrower strips (i.e. by making δx smaller). If we allow $\delta x \to 0$ then the area in equation 1 approaches a limit which is equal to the exact area beneath the curve.

$$\text{Area} = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\delta x$$

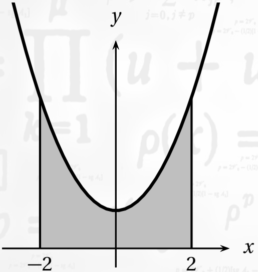
This equation can be shown to be the same as the definite integral

$$\boxed{\text{Area} = \int_a^b f(x)\,\mathrm{d}x} \qquad (2)$$

Area beneath a curve (above the x-axis)

Calculate the area bounded by the curve $y = 3x^2 + 1$, the lines $x = -2$, $x = 2$ and the x-axis.

A picture of the defined area is shown below.



In this case we can see that no part of the curve crosses the x-axis so the area can be found by evaluating the integral in one step.

$$\begin{aligned} \text{Area} &= \int_{-2}^{2} (3x^2 + 1)\,\mathrm{d}x \\ &= \left[x^3 + x\right]_{-2}^{2} \\ &= \left(2^3 + 2\right) - \left((-2)^3 - 2\right) \\ &= (8 + 2) - (-8 - 2) \\ &= 10 + 10 \\ &= 20 \end{aligned}$$

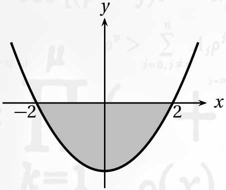
The exact area beneath the curve between the limits shown in the picture is 20 units^2.

It's useful to know what the graph looks like in case any of the area being considered is on the other side of the x-axis.

Area beneath a curve (below the x-axis)

Calculate the area bounded by the curve $y = x^2 - 4$ and the x -axis.

Here's a sketch showing the required area.



$$\begin{aligned}\text{Area} &= \int_{-2}^{2} (x^2 - 4)\,\mathrm{d}x \\ &= \left[\tfrac{1}{3}x^3 - 4x\right]_{-2}^{2} \\ &= \left(\tfrac{2^3}{3} - 4(2)\right) - \left(\tfrac{(-2)^3}{3} - 4(-2)\right) \\ &= \left(\tfrac{8}{3} - 8\right) - \left(-\tfrac{8}{3} + 8\right) \\ &= \tfrac{8}{3} + \tfrac{8}{3} - 8 - 8 \\ &= -\frac{32}{3}\end{aligned}$$

A loop of the curve encloses an area below the axis - this is what we must find.

Solve the equation $x^2 - 4 = 0$ to find the intersection with the x-axis.

The solutions ($x = -2$, $x = 2$) are the limits of the integral.

The negative value occurs because the area is beneath the x-axis. However, we do not give a sign when dealing with area so we can ignore it.

The area bounded by the curve and the x-axis is $\frac{32}{3}$ units^2.

Dealing with "negative" area

In the previous example we had to evaluate

$$\text{Area} = \int_{-2}^{2} (x^2 - 4)\,\mathrm{d}x$$

The graph indicated that the area was below the axis (negative y-values) and so we might have anticipated that the integral would have a negative sign.

It is better to force the integral to give a positive value — because area is defined to be a positive quantity.

We could have written

$$\text{Area} = \int_{2}^{-2} (x^2 - 4)\,\mathrm{d}x$$

Try it! We can swap the limits to ensure a positive value.

The alternative is to use existing limit order and take the *absolute value* of the definite integral.

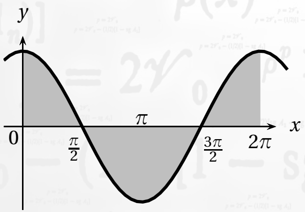
Using the area integral correctly

The integration formula for calculating areas:

$$\text{Area} = \int_a^b f(x)\,\mathrm{d}x$$

will give the *net* area beneath a curve.This means if some of the area is below the x-axis (negative y values) then this will subtract from the area above the x-axis.

Consider the area bounded by the curve $y = \cos x$ and the coordinate axes over the interval $0 \le x \le 2\pi$. Here is the picture:



Simply using the limits 0 to 2π will not give the correct area!Try it:

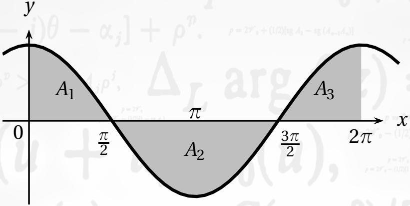
$$\int_0^{2\pi} \cos x\,\mathrm{d}x = [\sin x]_0^{2\pi} = \sin 2\pi = 0$$

But it is clear from the graph that the area is not zero!

The total shaded area above the x-axis is identical to the amount of area below it. It cancels out to give zero net area when we do the integral.

Using the area integral correctly

Simply using the limits 0 to 2π will not give the correct area. Instead we must find the area of each region separately.



$$\text{Area} = A_1 + A_2 + A_3$$

Integrating to find A_2 will give a negative value (because y is negative). Just ignore the sign of the value (by taking the modulus, or by reversing the order of the limits).

$$\begin{aligned}
\text{Area} &= \int_0^{\pi/2} \cos x \, dx + \left| \int_{\pi/2}^{3\pi/2} \cos x \, dx \right| + \int_{3\pi/2}^{2\pi} \cos x \, dx \\
&= [\sin x]_0^{\pi/2} + \left| [\sin x]_{\pi/2}^{3\pi/2} \right| + [\sin x]_{3\pi/2}^{2\pi} \\
&= 1 + |-2| + 1 = 4
\end{aligned}$$

So the total area is actually 4 units^2.

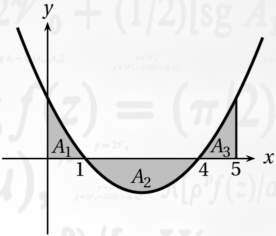
Areas above and below the x-axis

Calculate the area bounded by the curve $y = x^2 - 5x + 4$, the coordinate axes and the line $x = 5$.

Sketch the function so that the limits of integration can be found.

The intersection with the axis can be found by solving $x^2 - 5x + 4 = 0$.

It factorises to give $(x-1)(x-4) = 0$ so that $x = 1$ and $x = 4$.



The areas have to be treated separately, Area $= A_1 + A_2 + A_3$

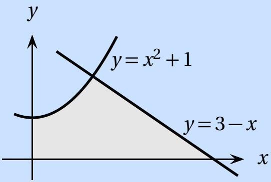
$$\text{Area} = \int_0^1 x^2 - 5x + 4\,dx + \int_4^1 x^2 - 5x + 4\,dx + \int_4^5 x^2 - 5x + 4\,dx$$

The limits of A_2 were reversed to give a positive value.

$$\begin{aligned}\text{Area} &= \left[\tfrac{1}{3}x^3 - \tfrac{5}{2}x^2 + 4x\right]_0^1 + \left[\tfrac{1}{3}x^3 - \tfrac{5}{2}x^2 + 4x\right]_4^1 + \left[\tfrac{1}{3}x^3 - \tfrac{5}{2}x^2 + 4x\right]_4^5 \\ &= \left(\tfrac{11}{6} - 0\right) + \left(\tfrac{11}{6} - \left(-\tfrac{8}{3}\right)\right) + \left(-\tfrac{5}{6} - \left(-\tfrac{8}{3}\right)\right) = \tfrac{11}{6} + \tfrac{9}{2} + \tfrac{11}{6} \\ \therefore \text{Area} &= \frac{49}{6}\ \text{units}^2\end{aligned}$$

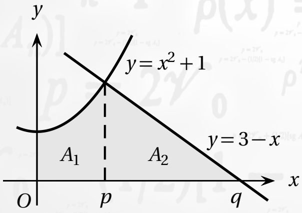
Area beneath two different curves

Consider the curves $y = x^2 + 1$ and the line $y = 3 - x$, which are shown on the graph.



Calculate the total area of the shaded region beneath the curves.

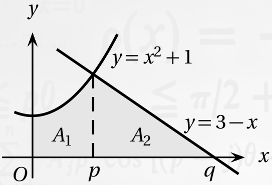
This area here is bounded by different curves. To find the total area we should break the region into subregions beneath each curve.



The total area is $A_1 + A_2$:

$$\text{Area} = \int_0^p (x^2 + 1)\,\mathrm{d}x + \int_p^q (3 - x)\,\mathrm{d}x$$

We must calculate the limits for each integral — the x values of p and q.



The line and curve intersect at p so we solve:

$$x^2+1=3-x$$

Therefore:

$$\begin{aligned} x^2+x-2 &= 0 \\ (x+2)(x-1) &= 0 \end{aligned}$$

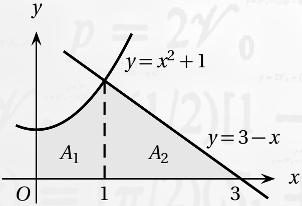
So $x=1$ and $x=-2$. From the picture it's clear that the limit must be $p=1$.

The line meets the x-axis when $y=0$ so

$$3-x=0 \quad \therefore x=3$$

So the other limit is $q=3$.

Now that the limits are known, the total area can be computed. The completed graph is shown below.



$$\begin{aligned} \text{Area} &= \int_0^1 (x^2+1)\,\mathrm{d}x + \int_1^3 (3-x)\,\mathrm{d}x \\ &= \left[\tfrac{1}{3}x^3+x\right]_0^1 + \left[3x-\tfrac{1}{2}x^2\right]_1^3 \\ &= \tfrac{4}{3}+2 \\ &= \tfrac{10}{3} \end{aligned}$$

Summary

- The area enclosed by a curve $y = f(x)$ and the limits $x = a$ and $x = b$ can be calculated using the definite integral

$$\boxed{\text{Area} = \int_a^b f(x)\,\mathrm{d}x}$$

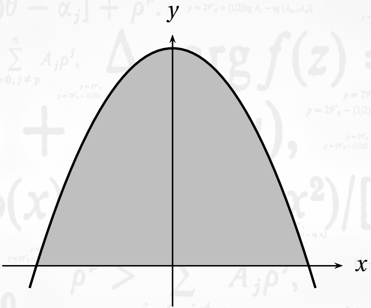
  Provided the area is completely above (or completely below) the x-axis then nothing more needs to be considered.

- If the area is partially above and below the x-axis then the above integral will only give the *net* area. The total area will need to found by considering positive and negative regions separately.
- If limits are not given then they will need to be calculated; e.g. by solving $f(x) = 0$.
- If the area is defined by more than one function — each region must be calculated seperately.

Test yourself...

Solve the following four area problems using integration.

1. The area enclosed by the curve $y = 16 - x^2$ and the x-axis.

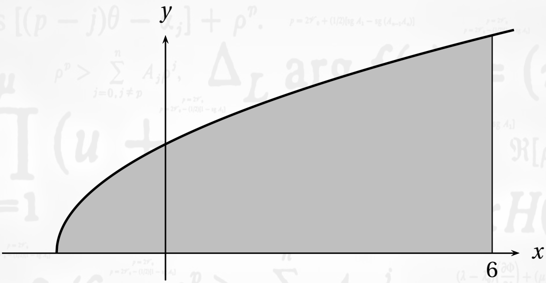


Answers:

1. Area is $\frac{256}{3}$.

Test yourself...

2. The area enclosed by the curve $y = \sqrt{2x+4}$, the x-axis and the line $x = 6$.

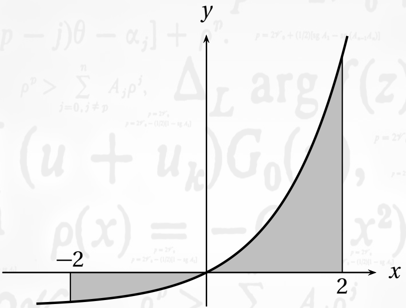


Answers:

2. Area is $\frac{64}{3}$.

Test yourself...

3. The area enclosed by the curve $y = e^x - 1$, the x-axis and the lines $x = -2$ and $x = 2$. Give your answer in terms of e.



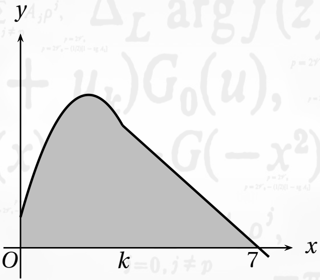
Answers:

3. Area is $e^2 + \frac{1}{e^2} - 2$.

4. The picture below a shaded area bounded by the curve

$$f(x) = \begin{cases} 1+4x-x^2 & 0 \leq x < k \\ 7-x & k \leq x < 7 \end{cases}$$

Find the value of k and hence calculate the exact value of the shaded region.



Answers:

4. Area is 20.