Integration by parts

Calculus 8

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Integration by Parts

Recall the product rule for differentiation. Given two functions $u(x)$ and $v(x)$ we find the derivative from this:

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

We're going to find an equivalent rule for *integrating* products. Rearrange to obtain

$$u\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(uv) - v\frac{\mathrm{d}u}{\mathrm{d}x}$$

Integrate both sides of this equation

$$\begin{aligned}\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x &= \int\left(\frac{\mathrm{d}}{\mathrm{d}x}(uv) - v\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x \\ &= \int \frac{\mathrm{d}}{\mathrm{d}x}(uv)\,\mathrm{d}x - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x\end{aligned}$$

Look at the first term on the LHS; integrating the derivative gives uv.

$$\boxed{\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x}$$

This rule is called integration by parts. The constant of integration is missing from this equation but it appears, as usual, in the answer.

Integration by parts - simple case

Use integration by parts to find $\int 3x\cos x\,\mathrm{d}x$

We have to choose which function to differentiate and which to integrate. Let's choose

$$u = 3x \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$$

Differentiating u and integrating $\frac{\mathrm{d}v}{\mathrm{d}x}$ gives:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 3 \qquad \text{and} \qquad v = \sin x$$

Substituting these four expressions into the formula $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$

$$\int 3x\cos x\,\mathrm{d}x = (3x)(\sin x) - \int \sin x(3)\,\mathrm{d}x$$
$$= 3x\sin x - \int 3\sin x\,\mathrm{d}x$$

We must still integrate part of the RHS - but this time it is easy. Finishing the integration (and including the constant):

$$\int 3x\cos x\,\mathrm{d}x = 3x\sin x - (-3\cos x) + C$$
$$= 3x\sin x + 3\cos x + C$$

The choice of u and $\frac{\mathrm{d}v}{\mathrm{d}x}$ at the start is very important. What would have happened if we had swapped our choices in this example?

Speeding things up...

The integration by parts formula is

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

A more compact way to write this formula is:

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

We'll use this notation in the remaining examples to speed up the working out.

Incorporating the chain rule

Use integration by parts to find $\int 5x\mathrm{e}^{-3x}\,\mathrm{d}x$

We have to choose which function to differentiate and which to integrate. Let's choose

$$u = 5x \quad \text{and} \quad \mathrm{d}v = \mathrm{e}^{-3x}$$

Differentiating u and integrating $\frac{\mathrm{d}v}{\mathrm{d}x}$ gives:

$$\mathrm{d}u = 5 \quad \text{and} \quad v = -\tfrac{1}{3}\mathrm{e}^{-3x}$$

Substituting these four expressions into the formula $\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$

$$\begin{aligned}
\int 5x\mathrm{e}^{-3x}\,\mathrm{d}x &= 5x(-\tfrac{1}{3}\mathrm{e}^{-3x}) - \int (-\tfrac{1}{3}\mathrm{e}^{-3x})(5)\,\mathrm{d}x \\
&= -\tfrac{5}{3}x\mathrm{e}^{-3x} + \tfrac{5}{3}\int \mathrm{e}^{-3x}\,\mathrm{d}x \\
&= -\tfrac{5}{3}x\mathrm{e}^{-3x} + \tfrac{5}{3}(-\tfrac{1}{3}\mathrm{e}^{-3x}) + C \\
&= -\tfrac{5}{3}x\mathrm{e}^{-3x} - \tfrac{5}{9}\mathrm{e}^{-3x} + C
\end{aligned}$$

Practice differentiating with the chain rule *and* integrating with the reverse chain rule.

Evaluating definite integrals

Evaluate the integral $\int_1^3 \frac{\ln x}{x^2}\,\mathrm{d}x$

We don't have a derivative for $\ln x$ so we a forced to choose:

$$u = \ln x \quad \text{and} \quad \mathrm{d}v = \frac{1}{x^2}$$

Differentiating u and integrating $\frac{\mathrm{d}v}{\mathrm{d}x}$ (with the reverse chain rule) gives:

$$\mathrm{d}u = \frac{1}{x} \quad \text{and} \quad v = -\frac{1}{x}$$

Substituting these four expressions into the formula $\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$

$$\begin{aligned}
\int \frac{\ln x}{x^2}\mathrm{d}x &= \ln x\left(-\frac{1}{x}\right) - \int\left(-\frac{1}{x}\right)\frac{1}{x}\,\mathrm{d}x \\
&= -\frac{\ln x}{x} + \int \frac{1}{x^2}\,\mathrm{d}x \\
\therefore \int_1^3 \frac{\ln x}{x^2}\mathrm{d}x &= \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^3 \\
&= \left(-\frac{\ln 3}{3} - \frac{1}{3}\right) - \left(-\frac{\ln 1}{1} - 1\right) = \tfrac{1}{3}(2 - \ln 3)
\end{aligned}$$

By Parts - repeated application

Find $\int x^2 \sin x \,\mathrm{d}x$

We will choose

$$u = x^2 \qquad \text{and} \qquad \mathrm{d}v = \sin x$$

Differentiating u and integrating $\frac{\mathrm{d}v}{\mathrm{d}x}$ gives:

$$\mathrm{d}u = 2x \qquad \text{and} \qquad v = -\cos x$$

Putting these into the 'By Parts' formula gives

$$\begin{aligned} \int x^2 \sin x \,\mathrm{d}x &= -x^2 \cos x - \int (-2x \cos x)\,\mathrm{d}x \\ &= -x^2 \cos x + 2 \int x \cos x \,\mathrm{d}x \qquad (1) \end{aligned}$$

To integrate $x \cos x$ we have to apply integration by parts again.

Let:

$$u = x \qquad \text{and} \qquad dv = \cos x$$

$$du = 1 \qquad \text{and} \qquad v = \sin x$$

Using the integration by parts formula we have

$$\int x\cos x\,dx = x\sin x - \int (1)\sin x\,dx$$
$$= x\sin x - \int \sin x\,dx$$
$$= x\sin x + \cos x$$

Put this result back into equation (1) and add the constant of integration.

$$\int x^2 \sin x\,dx = -x^2\cos x + 2(x\sin x + \cos x) + C$$

Choosing u and dv

At this point, with a few examples done, we'll discuss strategies for making good choices at the start of the method.

When it comes to choosing u, choose whichever function comes first in this list:

- Log function $\ln x$
- Inverse trig function
- Algebraic function (e.g. ax^n)
- Trig or exponential functions

But be aware that these guidelines don't cover every possible integral you'll see. Some more general advice about choosing:

- Let dv be the most complicated portion of the integrand that can be *easily* integrated.
- Let u be that portion of the integrand whose derivative du is a 'simpler' function than u itself.

The 'by parts' method will become easier in time. Eventually, your own experience will begin to inform your decisions in these matters.

Let's look at good choices for the 'by parts' method. We won't do the work here: just the initial choice.

- $\int x^3 e^{2x}\,dx$. Choose $u = x^3$ and $dv = e^{2x}$
- $\int x \ln x\,dx$. Choose $u = \ln x$ and $dv = x$
- $\int 2x \tan^{-1} x\,dx$. Choose $u = \tan^{-1} x$ and $dv = 2x$
- $\int x\sqrt{x+3}\,dx$. Choose $u = x$ and $dv = \sqrt{x+3}$

If the choice is not obvious then a good way to tell whether you made a bad choice is that the second integration is more difficult (more complex) than the one you started with.

The integration by parts 'trick'!

Find $\int e^x \sin x \, dx$

Neither of these functions gives a 'simpler' one after differentiation. Let's put

$$u = e^x \quad \text{and} \quad dv = \sin x$$

There was nothing special about this choice. From this we get:

$$du = e^x \quad \text{and} \quad v = -\cos x$$

The 'By parts' formula gives:

$$\begin{aligned} \int e^x \sin x \, dx &= -e^x \cos x - \int e^x(-\cos x) \, dx \\ &= -e^x \cos x + \int e^x \cos x \, dx \end{aligned} \tag{2}$$

Now we must integrate $e^x \cos x$ by parts.

$$u = e^x \quad \text{and} \quad dv = \cos x$$

$$du = e^x \quad \text{and} \quad v = \sin x$$

Using the formula and substituting back into the original integral 2 we get:

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

$$\int e^x \sin x \,dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \,dx$$

It seems that we have made no progress at all! However, notice that the integral on the LHS also appears on the RHS. This means we can combine them!

By adding $\int e^x \sin x \,dx$ to both sides of the above equation get:

$$2\int e^x \sin x \,dx = -e^x \cos x + e^x \sin x$$

By dividing both sides by 2 (and then including the constant of integration)

$$\int e^x \sin x \,dx = \frac{-e^x \cos x + e^x \sin x}{2} + C$$

We seem to have obtained the answer very easily!

Useful advice...

In the previous example, the initial choice of u and dv was not important. However, *the choice for the second integration was important.* If the initial choice was to integrate the trig function then you must do the same again on the second integral. If you don't, you'll simply prove that $0 = 0$. (Try it!)

Integrating $\ln x$

Consider the integral $\int \ln x \, dx$.

We can use integration by parts to find the integral of the natural logarithm function by regarding it as a product $1 \times \ln x$. Let:

$$u = \ln x \qquad \text{and} \qquad dv = 1$$

We are forced to make this choice because we don't know how to integrate $\ln x$. Therefore:

$$du = \frac{1}{x} \qquad \text{and} \qquad v = x$$

Using the integration by parts formula:

$$\begin{aligned}
\int (1) \ln x \, dx &= x \ln x - \int x \left(\frac{1}{x} \right) dx \\
&= x \ln x - \int 1 \, dx \\
&= x \ln x - x + C \\
\int \ln x \, dx &= x(\ln x - 1) + C
\end{aligned}$$

This method can be used to integrate other functions, like the inverse trig functions ($\sin^{-1} x$, etc).

Test yourself...

Try the following problems. They can all be integrated by parts.

❶ $\int x e^x \, dx$

❷ $\int x^2 \cos 2x \, dx$

❸ $\int_0^1 4x\sqrt{1+8x} \, dx$

❹ $\int_0^{\frac{\pi}{2}} e^{2x} \cos x \, dx$

Answers:

❶ $e^x(x-1)+C$

❷ $\frac{1}{2}x^2 \sin 2x + \frac{1}{2}x \cos 2x - \frac{1}{4}\sin 2x + C$

❸ $\frac{149}{30}$

❹ $\frac{1}{5}(e^{\pi}-2)$