Optimisation

Calculus 5

INU0115/515 (Maths 2)

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Introduction

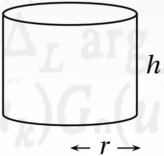
Scientists and engineers are often interested in optimisation. This means finding the best solution to a problem where there is a constraint on one or more variables. Some simple examples of this might include:

- Building a container of maximum volume from a given quantity of metal.
- Minimising the cost of construction of an object.
- Constructing a fence with a given length to enclose the greatest possible area.
- Finding the maximum displacement of a particle moving with simple harmonic motion.
- Finding the maximum height of a projectile moving under the action of gravity and air resistance.

In these examples we can use the methods seen previously for calculating *stationary points* to identify the best solutions and to prove that the solutions correspond to maximum or minimum values.

Maximising the volume of a cylinder

A factory is to manufacture cylindrical cans for its products. Given that the surface area is to be 600cm^2, what value must the radius take to give the maximum volume for the cylinder?



Here are some useful formulae for a cylinder. The volume V is given by

$$V = \pi r^2 h \tag{1}$$

where r is the radius and h is the height. The total surface area A of a cylinder is given by

$$A = 2\pi rh + 2\pi r^2 \tag{2}$$

The problem requires us to find a maximum value for V. We will use differentiation to look for a stationary point of the function and show that it represents a maximum value.

Maximising the volume of a cylinder

A factory is to manufacture cylindrical cans for its products. Given that the surface area is to be 600 cm^2, what value must the radius take to give the maximum volume for the cylinder? Calculate the maximum volume.

The formula for V contains two variables. We must eliminate one of them before we can differentiate. Since the surface area is constrained to be 600cm^2, then:

$$2\pi rh + 2\pi r^2 = 600$$

Rearrange to get h

$$\begin{aligned} \pi rh + \pi r^2 &= 300 \\ \pi rh &= 300 - \pi r^2 \\ h &= \frac{300 - \pi r^2}{\pi r} \end{aligned}$$

Substitute this into the formula for volume (equation 1):

$$V = \pi r^2 h = \pi r^2 \left(\frac{300 - \pi r^2}{\pi r} \right) = 300r - \pi r^3$$

Now we can differentiate to find stationary points.

We saw that

$$V = 300r - \pi r^3$$

Differentiate with respect to r:

$$\frac{\mathrm{d}V}{\mathrm{d}r} = 300 - 3\pi r^2$$

Stationary points occur when $\frac{\mathrm{d}V}{\mathrm{d}r} = 0$, so we solve

$$\begin{aligned} 300 - 3\pi r^2 &= 0 \\ 3\pi r^2 &= 300 \\ \pi r^2 &= 100 \\ r &= \frac{10}{\sqrt{\pi}} \end{aligned}$$

We take the positive root because r represents a scalar (length).

Use the second derivative to test the stationary point.

$$\frac{\mathrm{d}^2 V}{\mathrm{d}r^2} = -6\pi r$$

This is negative for all positive values of r so it corresponds to a local maximum.

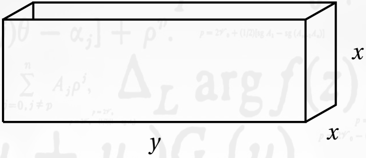
Therefore the volume of the can is maximised when $r = 5.64$ cm (2 d.p).Using this value we find that $h = 11.28$ cm and $V = 1128\ \mathrm{cm}^3$.

Procedure for finding maximum or minimum values

- Write down all the information given in the problem as formulae or equations.
- Identify the function/formula to be differentiated.
- Use substitution to express the function as a single variable.
- Find the stationary point of the function
- Use a test to verify it is a maximum or minimum point.

Maximising the volume of a box

Consider a metal open topped box with height and width of x metres and length y metres. Given that the area of metal used to construct the box is $54\,\mathrm{m}^2$, find the value of x which maximises the volume of the container.



The volume of the box:

$$V = x^2 y$$

The surface area of the box:

$$\begin{aligned} A &= x^2 + x^2 + xy + xy + xy \\ &= 2x^2 + 3xy \end{aligned}$$

And we are given that $2x^2 + 3xy = 54$.

We have to find stationary values of V so we have to express it in terms of x. So we eliminate y using the area relation:

$$\begin{aligned} 3xy &= 54 - 2x^2 \\ \therefore y &= \frac{54 - 2x^2}{3x} \end{aligned}$$

The volume of the box can be expressed in terms of x

$$\begin{aligned} V &= x^2 y \\ &= x^2\left(\frac{54-2x^2}{3x}\right) \\ \therefore V &= 18x - \tfrac{2}{3}x^3 \end{aligned}$$

Now differentiate:

$$\frac{\mathrm{d}V}{\mathrm{d}x} = 18 - 2x^2$$

Stationary points occur when the gradient is zero, i.e. when $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$, so

$$\begin{aligned} 18 - 2x^2 &= 0 \\ 18 &= 2x^2 \\ 9 &= x^2 \\ 3 &= x \end{aligned}$$

The second derivative test gives:

$$\frac{\mathrm{d}^2 V}{\mathrm{d}x^2} = -4x$$

This expression is negative when $x = 3\mathrm{m}$, showing that it corresponds to a local maximum.

A minimisation problem

A cylindrical can is to hold $20\pi \text{m}^3$. The material for the top and bottom costs £10 per m^2 and material for the side costs £8 per m^2. Find the radius r and height h of the most economical can and calculate its cost.

The volume of this cylinder is

$$V = \pi r^2 h$$

and we are given a constraint for the volume:

$$\pi r^2 h = 20\pi$$

The surface area of the top and bottom is

$$A_1 = \pi r^2 + \pi r^2 = 2\pi r^2$$

The surface area of the side is

$$A_2 = 2\pi r h$$

This time we have to minimise the *cost* whilst keeping the volume constant. The total cost C is given by:

$$\begin{aligned} C &= 10A_1 + 8A_2 \\ &= 20\pi r^2 + 16\pi r h \end{aligned}$$

We must eliminate h from this. Since $h = 20/r^2$:

$$\begin{aligned} C &= 20\pi r^2 + 16\pi r\left(\frac{20}{r^2}\right) \\ \therefore C &= 20\pi r^2 + \frac{320\pi}{r} \end{aligned}$$

We found that the cost of the can is given by

$$C = 20\pi r^2 + 320\pi r^{-1}$$

The gradient function is

$$\begin{aligned}\frac{\mathrm{d}C}{\mathrm{d}r} &= 40\pi r - 320\pi r^{-2} \\ &= 40\pi r - \frac{320\pi}{r^2}\end{aligned}$$

The stationary values occur where $\frac{\mathrm{d}C}{\mathrm{d}r} = 0$, so we solve:

$$40\pi r - \frac{320\pi}{r^2} = 0$$

Cancel π and cross multiply by r^2 to remove the fraction:

$$\begin{aligned}40r^3 - 320 &= 0 \\ r^3 &= 8\end{aligned}$$

Take the cube root to get $r = 2\text{m}$.

Use the second derivative test to classify the stationary point:

$$\frac{\mathrm{d}^2 C}{\mathrm{d}r^2} = 40\pi + 640\pi r^{-3} = 40\pi + \frac{640\pi}{r^3}$$

This quantity is positive for $r = 2$, confirming that it is a minimum value.

The total cost of the can is therefore $C = £753.98$.

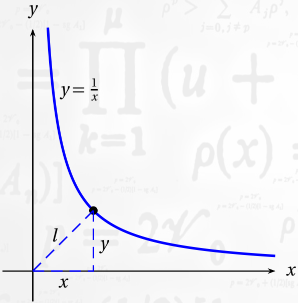
A more complicated problem

Given the curve

$$y = \frac{1}{x},\ x > 0$$

Calculate the minimum distance of this curve from the origin.

Here is a graph of the situation.



The picture shows a point on the curve with coordinates (x, y).

The distance of a point on the curve from the origin is given by Pythagoras's theorem to be

$$l = \sqrt{x^2 + y^2}$$

and since $y = \frac{1}{x}$ we can write this as:

$$l = \sqrt{x^2 + \left(\frac{1}{x}\right)^2} = \left(x^2 + \frac{1}{x^2}\right)^{\frac{1}{2}}$$

This is the function we need to minimise.

Differentiate l with the chain rule:

$$\frac{\mathrm{d}l}{\mathrm{d}x} = \frac{1}{2}\left(x^2+\frac{1}{x^2}\right)^{-\frac{1}{2}} \times \left(2x-\frac{2}{x^3}\right) = \frac{x-\frac{1}{x^3}}{\sqrt{x^2+\frac{1}{x^2}}}$$

We're looking for a minimum value for l so we must solve $\frac{\mathrm{d}l}{\mathrm{d}x}=0$:

$$\frac{x-\frac{1}{x^3}}{\sqrt{x^2+\frac{1}{x^2}}}=0$$

Multiply both sides by the denominator of the LHS:

$$x-\frac{1}{x^3}=0$$

Multiply both sides by x^3

$$\begin{aligned} x^4-1 &= 0 \\ x^4 &= 1 \\ x^2 &= \pm 1 \end{aligned}$$

So $x^2=-1$ (no real solutions) or $x^2=1$, which leads to $x=\pm 1$.

So there is a stationary value at $x=1$ (since the question requires $x>0$).

When $x=1$ then $y=1$ and the minimum distance is $l=\sqrt{2}$.