STATIONARY VALUES

CALCULUS 4

INU0115/515 (MATHS 2)

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Stationary values

Differentiation provides a way of investigating the rate of change of functions. This is useful because we might want to know when a function is increasing or decreasing value or when the function reaches extreme values (such as maximum or minimum points).

Stationary values

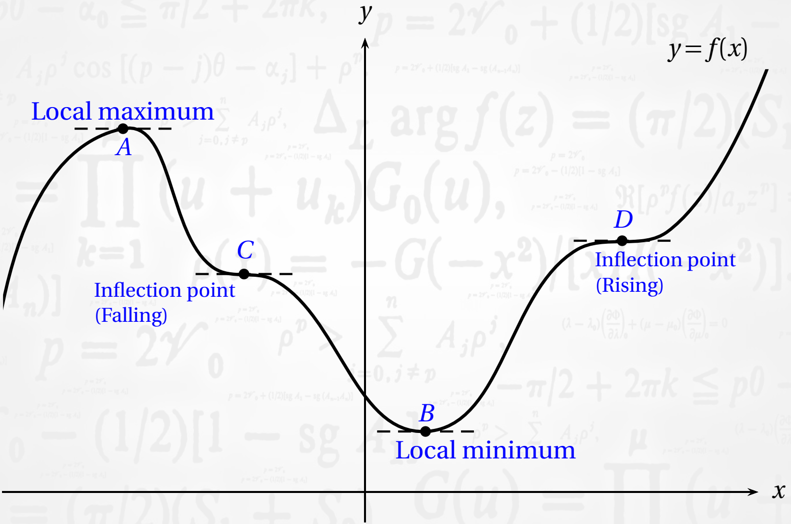
A stationary value of a function $y = f(x)$ is any value of the function at which its rate of change with respect to x is zero. This means stationary values of a function occur when

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

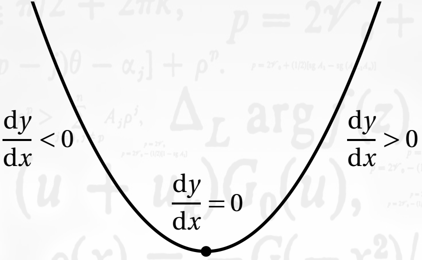
or in alternative notation when $y' = 0$, $f'(x) = 0$ or $\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)\right] = 0$.

Stationary values of a function can indicate maximum or minimum points ('turning points') on a graph.

Stationary points of a graph



Behaviour near a minimum point

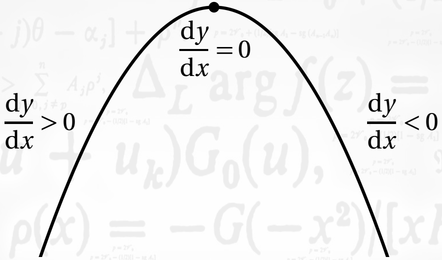


The gradient $\frac{dy}{dx}$ changes from negative to positive at the SP (as x increases).

Since $\frac{dy}{dx}$ is increasing it might be possible that $\frac{d^2y}{dx^2} > 0$ at the SP.

Also, the y-values to each side of the SP are greater than the y-value of the SP.

Behaviour near a maximum point



The gradient $\frac{dy}{dx}$ changes from positive to negative at the SP (as x increases).

Since $\frac{dy}{dx}$ is decreasing it might be possible that $\frac{d^2y}{dx^2} < 0$ at the SP.

Also, the y-values to each side of the SP are less than the y-value of the SP.

Three tests to classify the stationary point

1. Do the second derivative test at the SP

$$\frac{d^2y}{dx^2} < 0 \quad \Rightarrow \text{Local MAXIMUM}$$

$$\frac{d^2y}{dx^2} > 0 \quad \Rightarrow \text{Local MINIMUM}$$

$$\frac{d^2y}{dx^2} = 0 \quad \Rightarrow \text{TEST HAS FAILED!}$$

When this test fails[1] you must use another to classify the SP.

2. Examine the *sign* of $\frac{dy}{dx}$ to each side of the SP.
3. Examine the value of y to each side of the SP.

[1] It is possible for $\frac{d^2y}{dx^2}$ to be zero at a maximum, minimum or an inflection point.

Finding and classifying stationary points

Find and classify the stationary points of the graph of

$$y = x^2 - 8x + 1$$

Differentiate to get the gradient function

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 8$$

Stationary points occur where $y' = 0$ so we solve

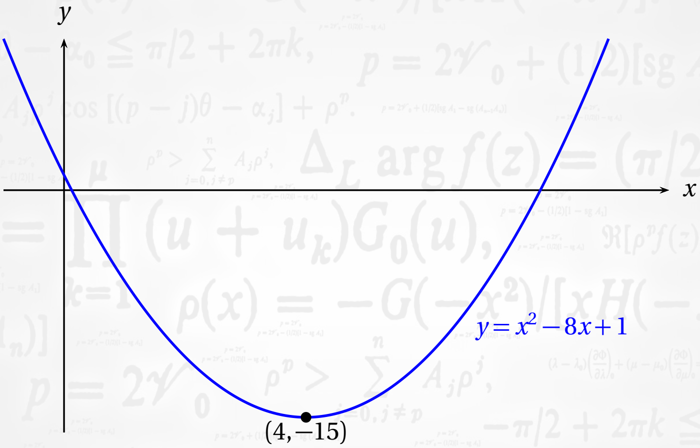
$$\begin{aligned} 2x - 8 &= 0 \\ 2x &= 8 \\ x &= 4 \end{aligned}$$

So the stationary point is $(4, -15)$.

Let's do the second derivative test.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2$$

Since the second derivative is positive, we conclude that (4,-15) is a local minimum of the curve.


y
x
$y = x^2 - 8x + 1$
(4, −15)

Finding and classifying stationary points

Find and classify the stationary points of the graph of

$$y = (11 - x)(1 + x)$$

Expand the brackets to get $y = 11 + 10x - x^2$.

Differentiate to get the gradient function

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 10 - 2x$$

Stationary points occur where $y' = 0$ so we solve

$$\begin{aligned} 10 - 2x &= 0 \\ 10 &= 2x \\ \therefore x &= 5 \end{aligned}$$

So the stationary point is (5,36).

Try the second derivative test.In this case $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -2$

Since the second derivative is negative, we conclude that $(5, 36)$ is a local maximum of the curve.

A more complicated second derivative test

Find and classify the stationary points of the curve given by

$$y = x^3 + x^2 - x + 1$$

Begin by differentiating:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 2x - 1$$

Stationary points occur where $y' = 0$ so we solve

$$\begin{aligned} 3x^2 + 2x - 1 &= 0 \\ (3x-1)(x+1) &= 0 \end{aligned}$$

So the stationary points occur at $x = \frac{1}{3}$ and $x = -1$.

The coordinates of the stationary points are $(\frac{1}{3}, \frac{22}{27})$ and $(-1, 2)$.

Try the second derivative test.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 6x + 2$$

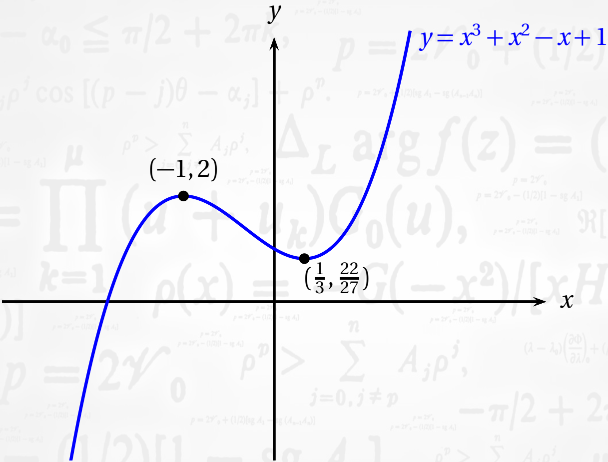
Substitute the x-values at the SP into this.

$$\left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x=\frac{1}{3}} = 4$$

So $(\frac{1}{3}, \frac{22}{27})$ is a local minimum point.

$$\left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x=-1} = -4$$

So $(-1, 2)$ is a local maximum point.


y
$y = x^3 + x^2 - x + 1$
$(-1, 2)$
$(\frac{1}{3}, \frac{22}{27})$
x

Alternative to the second derivative test

Find and classify the stationary points of the curve given by

$$y = (2x+1)^4$$

Find the gradient using the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 8(2x+1)^3$$

Stationary points occur where $y' = 0$ so we solve

$$\begin{aligned} 8(2x+1)^3 &= 0 \\ (2x+1)^3 &= 0 \\ 2x+1 &= 0 \end{aligned}$$

There is a stationary point at $x = -\frac{1}{2}$.

Try the second derivative test.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 48(2x+1)^2$$

Substitute the x-values at the SP into this.

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=-\frac{1}{2}} = 0$$

The test has failed; so we'll examine the gradient near the SP.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 8(2x+1)^3$$

We are only interested in the sign of the value (whether it is positive or negative).

x	-0.6	-0.5	-0.4
y'	-	0	+
Slope	\	—	/

The slope shows that the SP at $x = -\frac{1}{2}$ is a local minimum point.

Alternative to the second derivative test

Find and classify the stationary points of the curve given by

$$y = 4x^5 - x^4$$

Find the gradient

$$\frac{dy}{dx} = 20x^4 - 4x^3$$

Stationary points occur where $y' = 0$ so we solve

$$\begin{aligned} 20x^4 - 4x^3 &= 0 \\ 4x^3(5x - 1) &= 0 \end{aligned}$$

So the stationary points occur where $x = \frac{1}{5}$ and $x = 0$.

Try the second derivative test.

$$\frac{d^2y}{dx^2} = 80x^3 - 12x^2$$

Substitute the x-values at the SP into this.

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{5}} = \frac{4}{25}$$

So the SP at $x = \frac{1}{5}$ is a local minimum point.

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 0$$

The test has failed; we must try something else!

Alternative to the second derivative test

Find and classify the stationary points of the curve given by

$$y = 4x^5 - x^4$$

Since the second derivative test has failed, we will examine the gradient near the SP.

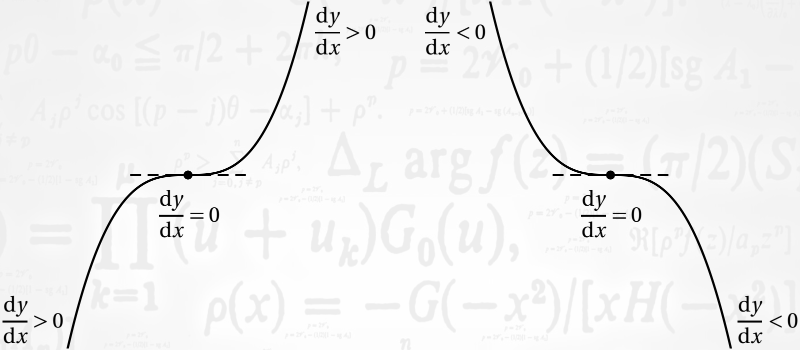
$$\frac{dy}{dx} = 20x^4 - 4x^3$$

We are only interested in the sign of the value (whether it is positive or negative).

x	-0.1	0	0.1
y'	+	0	-
Slope	/	—	\

The slope shows that the SP at $x = 0$ is a local maximum point.

Inflection points



Gradient $\frac{dy}{dx}$ *does not change sign* passing through the SP.

The y-value at the SP is between the y-values on each side of the SP.

The second derivative is zero $\left(\frac{d^2y}{dx^2} = 0\right)$ at an inflection point.

A curve with an inflection point?

Find and classify the stationary points of the curve given by

$$y = 2x^3 - 5$$

Start by finding the gradient:

$$\frac{dy}{dx} = 6x^2$$

Stationary points occur where $y' = 0$ so we solve

$$\begin{aligned} 6x^2 &= 0 \\ x &= 0 \end{aligned}$$

So there is a stationary point at (0,-5).

Try the second derivative test.

$$\frac{d^2y}{dx^2} = 12x$$

At the stationary point

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 0$$

The test has failed. So now we try something else...

A curve with an inflection point?

Find and classify the stationary points of the curve given by

$$y = 2x^3 - 5$$

Since the second derivative test has failed, we will examine the gradient near the SP.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2$$

We are only interested in the sign of the value (whether it is postive or negative).

x	-0.1	0	0.1
y'	+	0	+
Slope	/	—	/

The slope shows that the SP at $x = 0$ is a rising inflection point.

Another stationary point

Find and classify the stationary points of the function

$$f(x) = xe^{2x}$$

Begin by finding the first derivative. Using the product rule:

$$\begin{aligned} f'(x) &= 2xe^{2x} + e^{2x} \\ &= e^{2x}(2x+1) \end{aligned}$$

The stationary points occur where $f'(x) = 0$, so

$$e^{2x}(2x+1) = 0$$

Since $e^{2x} \neq 0$ the only solution to this equation is $x = -\frac{1}{2}$.

Try the second derivative test. Using the product rule:

$$\begin{aligned} f''(x) &= 2e^{2x} + (2x+1)(2)e^{2x} \\ &= 4e^{2x}(x+1) \end{aligned}$$

Therefore

$$f''(-\tfrac{1}{2}) = 4e^{-1}(\tfrac{1}{2}) = \frac{2}{e} > 0$$

The test has shown that the SP at $x = -\frac{1}{2}$ is a local minimum.

Stationary values of implicit functions

Find the stationary points of the function

$$x^2 + 2y^2 - 4x = 4$$

Differentiating with respect to x gives:

$$2x + 4y\frac{\mathrm{d}y}{\mathrm{d}x} - 4 = 0$$

Rearrange to get:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4-2x}{4y} = \frac{2-x}{2y}$$

Stationary points occur at where $y' = 0$:

$$\begin{aligned} \frac{2-x}{2y} &= 0 \\ 2-x &= 0 \\ \therefore x &= 2 \end{aligned}$$

Substitute into the original function:

$$\begin{aligned} 2^2 + 2y^2 - 4(2) &= 4 \\ 2y^2 &= 8 \\ y^2 &= 4 \\ y &= \pm 2 \end{aligned}$$

The stationary values are $(2, -2)$ and $(2, 2)$.

Classifying the stationary points requires further tests to be carried out.

It is often more complicated to carry out those tests. In the exam you will not have to do this.

Test yourself...

Find and classify the stationary values of the following functions.

1. $y = x^2 - 4x + 10$
2. $y = x^3 + 6x^2 + 9x - 5$
3. $y = x^3 + x + 1$.
4. $y = (x-2)^5$.

Answers:

1. Minimum at $(2, 6)$
2. Maximum at $(-3, -5)$, Minimum at $(-1, -9)$.
3. No stationary values.
4. Rising inflection at $(2, 0)$.