Logarithmic differentiation

Calculus 3

INU0115/515 (Maths 2)

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Recap: differentiating logarithmic functions

Remember the chain rule!

Find the derivative of the function $y = \ln \cos x$.

Let $u = \cos x$, which means that $y = \ln u$. Differentiating these:

$$\frac{du}{dx} = -\sin x \quad \text{and} \quad \frac{dy}{du} = \frac{1}{u}$$

Applying the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{\cos x} \times (-\sin x) = -\tan x$$

Differentiating logarithmic functions is a core part of the method you are about to see.

Example

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \ln\left(x^3 + 3\right)$

In this case

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^3+3} \times 3x^2 = \frac{3x^2}{x^3+3}$$

Example

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \ln(\sin x)$

In this case

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sin x} \times \cos x = \frac{\cos x}{\sin x} = \cot x$$

Ok...let's begin logarithmic differentiation.

Using logarithms to help with differentiation

Remember the following rules for logarithms?

$$\begin{aligned} \log(ab) &= \log a + \log b \\ \log\left(\frac{a}{b}\right) &= \log a - \log b \end{aligned}$$

These are useful for simplifying products and quotients.

It is easier to differentiate functions that are not products or quotients.

The power rule:

$$\log a^n = n \log a$$

will also be useful.

Logarithmic differentiation

Find the derivative of the function $y = x^2 e^x$ by taking logarithms.

Here we go! Take natural logs of both sides

$$\ln y = \ln(x^2 e^x)$$

Now separate the terms using the laws of logarithms:

$$\begin{aligned} \ln y &= \ln x^2 + \ln e^x \\ &= 2\ln x + x \ln e \\ \ln y &= 2\ln x + x \end{aligned}$$

Differentiate each side with respect to x:

$$\begin{aligned} \frac{1}{y}\frac{dy}{dx} &= \frac{2}{x} + 1 \\ \frac{dy}{dx} &= y\left(\frac{2}{x}+1\right) = x^2 e^x \left(\frac{2}{x}+1\right) \\ \therefore \frac{dy}{dx} &= x e^x (2+x) \end{aligned}$$

This problem could also be easily solved using the product rule.

Logarithmic differentiation

Given

$$y = \frac{x^4}{\tan x}$$

Use logarithms to find $\frac{\mathrm{d}y}{\mathrm{d}x}$

Take natural logs of both sides

$$\ln y = \ln\left(\frac{x^4}{\tan x}\right)$$

Using log laws:

$$\begin{aligned} \ln y &= \ln x^4 - \ln(\tan x) \\ &= 4\ln x - \ln(\tan x) \end{aligned}$$

Differentiate each side with respect to x:

$$\begin{aligned} \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4}{x} - \frac{\sec^2 x}{\tan x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= y\left(\frac{4}{x} - \frac{\sec^2 x}{\tan x}\right) = \frac{x^4}{\tan x}\left(\frac{4}{x} - \frac{\sec^2 x}{\tan x}\right) \end{aligned}$$

Derivative of a compound function

A curve is described by the equation

$$y = 3^x \cos x$$

Find the value of $\dfrac{dy}{dx}$ at the point where $x = \pi$.

Take natural logs of both sides:

$$\ln y = \ln(3^x \cos x)$$

Use the 'log rules' to expand the RHS:

$$\begin{aligned} \ln y &= \ln 3^x + \ln\cos x \\ &= x\ln 3 + \ln\cos x \end{aligned}$$

Differentiate each side:

$$\begin{aligned} \frac{1}{y}\frac{dy}{dx} &= \ln 3 + \frac{-\sin x}{\cos x} \\ &= \ln 3 - \tan x \end{aligned}$$

Therefore

$$\begin{aligned} \frac{dy}{dx} &= y(\ln 3 - \tan x) \\ &= 3^x \cos x(\ln 3 - \tan x) \end{aligned}$$

At $x = \pi$, we find that $\dfrac{dy}{dx} = -(3^\pi \ln 3) \approx -34.65$

Derivative of a compound function

Find the derivative of the function

$$y = \frac{x \sin x}{2x - 1}$$

Begin by taking natural logarithms of both sides.

$$\ln y = \ln\left(\frac{x \sin x}{2x - 1}\right)$$

Use the logarithm rules to separate the product and quotient:

$$\ln y = \ln x + \ln \sin x - \ln(2x - 1)$$

Now differentiate each side with respect to x.

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{\cos x}{\sin x} - \frac{2}{2x - 1}$$

$$\frac{dy}{dx} = y\left(\frac{1}{x} + \frac{\cos x}{\sin x} - \frac{2}{2x - 1}\right) = \frac{x \sin x}{2x - 1}\left(\frac{1}{x} + \cot x - \frac{2}{2x - 1}\right)$$

Test yourself...

Use logarithmic differentiation to answer the following problems.

1. Find $\frac{dy}{dx}$ when $y = x^2 \cos x$.
2. Find $\frac{dy}{dx}$ when $y = 10x^3 e^{2x} \tan x$.
3. Find $f'(x)$ when $f(x) = \frac{x \sec 2x}{e^x}$.

Answers:

1. $\frac{dy}{dx} = x^2 \cos x \left(\frac{2}{x} - \tan x \right)$
2. $\frac{dy}{dx} = 10x^3 e^{2x} \tan x \left(\frac{3}{x} + 2 + \frac{\sec^2 x}{\tan x} \right)$
3. $f'(x) = \frac{x \sec 2x}{e^x} \left(\frac{1}{x} + 2 \tan 2x - 1 \right)$