Implicit Differentiation

Calculus 3

INU0115/515 (Maths 2)

Dr Adrian Jannetta MIMA CMath FRAS

Explicit and implicit functions

Explicit functions

An explicit function is one where the link between the independent variable (say x) and the dependent variable (say y) is clearly defined.
Here are some examples of explicit functions:

$$y = x^2 + 2 \qquad y = 3\sin 2x + 2\cos 2x \qquad y = 5 + xe^{-x}$$

In each case, the value of y can be calculated from a value of x.
So far, we have only carried out differentiation of explicit functions.

Implicit functions

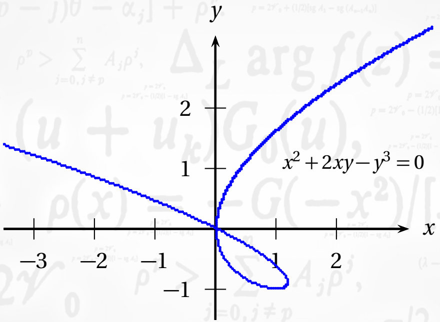
An implicit function is one where the relationship between the independent variable and dependent variable is *not* given in an explicit form.
For example:

$$x^2 + y^2 = 4 \qquad \sin y = e^{xy} \qquad xy^2 = 8x - \ln y$$

It is sometimes possible to rearrange an implicit function to obtain an explicit function. However, if we want to differentiate then we can do it using *implicit differentiation.*

Implicit functions

Implicit functions have the general form $F(x, y) = 0$. The LHS is read as 'function of x and y'. Implicit functions can describe some complicated curves!



Each place on the curve shown here has a well defined slope (gradient)
Therefore, we should be able to use calculus to figure out the gradient function $\frac{dy}{dx}$ for the implicit function.

Applying the chain rule to implicit functions

Consider the following two derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2) \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(y^2)$$

As we have seen, the first is just $2x$. Can the second be just $2y$? That would be the answer if we were differentiating with respect to y, but the $\mathrm{d}x$ tells us that that we are still differentiating with respect to x.

Since y^2 depends on x we have to use the chain rule.

Let $u = y^2$ so that

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^2) = \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}y} = 2y$$

Applying the chain rule we can write:

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^2) = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y} \times \frac{\mathrm{d}y}{\mathrm{d}x} = 2y\frac{\mathrm{d}y}{\mathrm{d}x}$$

This time the answer includes the gradient term $\frac{\mathrm{d}y}{\mathrm{d}x}$!

Implicit differentiation

Implicit differentiation

Given $F(y)$ where y is a function of x, then we can differentiate with respect to x using:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[F(y)\right] = \frac{\mathrm{d}}{\mathrm{d}y}\left[F(y)\right] \times \frac{\mathrm{d}y}{\mathrm{d}x}$$

So, differentiate with respect to y and then multiply by $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Here are some examples:

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^3) = \frac{\mathrm{d}}{\mathrm{d}y}(y^3) \times \frac{\mathrm{d}y}{\mathrm{d}x} = 3y^2\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin y) = \frac{\mathrm{d}}{\mathrm{d}y}(\cos y) \times \frac{\mathrm{d}y}{\mathrm{d}x} = \cos y\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(5\mathrm{e}^y) = \frac{\mathrm{d}}{\mathrm{d}y}(5\mathrm{e}^y) \times \frac{\mathrm{d}y}{\mathrm{d}x} = 5\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x}$$

Gradient of a circle

A circle is defined by the equation

$$x^2 + y^2 = 25$$

Find the gradient of the circle at the point (3,4).

We differentiate each term with respect to x

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

Remember the rule for differentiating y; it follows the usual rule but you must also multiply by $\frac{dy}{dx}$:

$$2x + 2y\frac{dy}{dx} = 0$$

Rearrange to get the gradient:

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

At the point (3,4) the gradient of the circle is $-\frac{3}{4}$.

Differentiating an implicit function

A curve is defined by the equation

$$x^3 + 2y^2 - 3x + 4y - 19 = 0$$

Find an expression for $\frac{dy}{dx}$.

Differentiating term by term

$$3x^2 + 4y\frac{dy}{dx} - 3 + 4\frac{dy}{dx} = 0$$

Factorise the terms with $\frac{dy}{dx}$:

$$\frac{dy}{dx}(4y+4) + 3x^2 - 3 = 0$$

Now rearrange to isolate the $\frac{dy}{dx}$ term:

$$\frac{dy}{dx} = \frac{3-3x^2}{4y+4}$$

The rules of differentiation

The rules of differentiation - the chain, product and quotient rules - work in the usual way. The only difference occurs when differentiating terms containing y, in which case we have to do it as shown in the previous examples.

Incorporating the chain rule

Find the gradient function of the curve which is described by

$$\ln(y^2+3)-2x^2=5$$

As before we differentiate term by term. The first must be found using the chain rule.

$$\frac{2y}{y^2+3}\frac{\mathrm{d}y}{\mathrm{d}x}-4x=0$$

Now rearrange to get the gradient:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{4x(y^2+3)}{2y}=\frac{2x(y^2+3)}{y}$$

Another example; the product rule

Consider the curve defined by

$$\sin y + 3x^2y^3 + 1 = 0$$

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for this curve.

Differentiate term by term. In this case we will need to use the product rule with the second term.

$$\cos y\frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2(3y^2\frac{\mathrm{d}y}{\mathrm{d}x}) + y^3(6x) = 0$$

Now simplify the terms

$$\cos y\frac{\mathrm{d}y}{\mathrm{d}x} + 9x^2y^2\frac{\mathrm{d}y}{\mathrm{d}x} + 6xy^3 = 0$$

Factorise:

$$\frac{\mathrm{d}y}{\mathrm{d}x}(\cos y + 9x^2y^2) + 6xy^3 = 0$$

and rearrange to get the gradient:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{6xy^3}{\cos y + 9x^2y^2}$$

Cutting down the working out...

In this type of work it is useful to use the 'prime' notation. You might find it easier (and quicker) to do these questions by writing y' instead of $\frac{dy}{dx}$.

Incorporating the quotient rule

Find an expression for the gradient for the curve defined by

$$2\tan y + \frac{3x}{5y^2} = 1$$

Differentiate term by term. The quotient rule is applied to the second term on the LHS.

$$(2\sec^2 y)y' + \frac{(5y^2)(3) - 3x(10yy')}{(5y^2)^2} = 0$$

Simplify:

$$\begin{aligned}(2\sec^2 y)y' + \frac{15y^2 - 30xyy'}{25y^4} &= 0 \\ (2\sec^2 y)y' + \frac{3y - 6xy'}{5y^3} &= 0\end{aligned}$$

Cross multiply by $5y^3$:

$$(10y^3\sec^2 y)y' + 3y - 6xy' = 0$$

Now factorise

$$(10y^3\sec^2 y - 6x)y' + 3y = 0$$

Rearrange to get the gradient:

$$y' = -\frac{3y}{10y^3\sec^2 y - 6x}$$

Test yourself...

You should be able to solve the following problems if you have understood everything in these notes.

1. Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(y^2+\mathrm{e}^{2y}\right)$
2. Given the curve $x^2-y^2+7x=2$, find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.
3. Given the curve $2x^3y^2-\sin y=0$, find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.
4. Find the equation of the normal to the curve $x^2-4xy+y^2=24$ at the point $(2,10)$.

Answers:

1. $2\frac{\mathrm{d}y}{\mathrm{d}x}(y+\mathrm{e}^{2y})$
2. $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x+7}{2y}$
3. $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{6x^2y^2}{4x^3y-\cos y}$
4. Normal equation is $y=-\frac{1}{3}x+\frac{32}{3}$