PRODUCT & QUOTIENT RULES

CALCULUS 2

INU0115/515 (MATHS 2)

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Objectives

In this presentation we'll continue learning how to differentiate more complicated functions.

There are rules for differentiating products of the form

$$y = u(x) \times v(x)$$

and quotients of the form

$$y = \frac{u(x)}{v(x)}$$

After some simple examples we'll practice incorporating the chain rule into some more complicated products and quotients.

The product rule

To differentiate a function consisting of products, like these:

$$x \sin x \qquad \ln x \cos x \qquad 3x^2 e^x$$

we must use a method called the product rule.

The Product Rule

Given a function of the form

$$y = uv$$

where u and v are both functions of x,then the derivative is given by

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Or in a shorter form:

$$y' = uv' + vu'$$

Product rule — derivation

Consider the function

$$y = u(x)v(x)$$

If δx is a small increase in x then there will be a corresponding increase in the values of u, v and y.

$$y + \delta y = (u + \delta u)(v + \delta v)$$

Expanding the brackets on the RHS:

$$\begin{aligned} y + \delta y &= u(v + \delta v) + \delta u(v + \delta v) \\ &= uv + u\delta v + v\delta u + \delta u\delta v \end{aligned}$$

If we subtract y from both sides (remembering that $y = uv$) we get:

$$\begin{aligned} \delta y &= u(v + \delta v) + \delta u(v + \delta v) - y \\ &= uv + u\delta v + v\delta u + \delta u\delta v - uv \\ \delta y &= u\delta v + v\delta u + \delta u\delta v \end{aligned}$$

Dividing both sides by δx we obtain:

$$\begin{aligned} \frac{\delta y}{\delta x} &= \frac{u\delta v + v\delta u + \delta u\delta v}{\delta x} \\ &= \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x} + \frac{\delta u\delta v}{\delta x} \\ &= u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u}{\delta x}\delta v \qquad (1) \end{aligned}$$

If we let $\delta x \to 0$ then also $\delta u \to 0$, $\delta v \to 0$ and the third term on the RHS vanishes. The following small changes become exact:

$$\lim_{\delta x \to 0} \frac{\delta v}{\delta x} = \frac{\mathrm{d}v}{\mathrm{d}x}, \lim_{\delta x \to 0} \frac{\delta u}{\delta x} = \frac{\mathrm{d}u}{\mathrm{d}x}, \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{\mathrm{d}y}{\mathrm{d}x}$$

Equation (1) becomes

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}}$$

Using the product rule

Use the product rule to find the first derivative of the function $y = x^2 \sin x$
.

The RHS is composed of two functions. Let

$$u = x^2 \quad \text{and} \quad v = \sin x$$

Differentiating each of these with respect to x we find:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$$

The product rule says

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = x^2 \cos x + 2x \sin x$$

Product rule

Given $f(x) = 4xe^x$, find $f'(x)$.

Let

$$u = 4x \qquad \text{and} \qquad v = e^x$$

Differentiate each to get:

$$\frac{du}{dx} = 4 \qquad \text{and} \qquad \frac{dv}{dx} = e^x$$

The product rule:

$$f'(x) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\therefore f'(x) = 4xe^x + 4e^x$$

There are common factors here:

$$f'(x) = 4e^x(x+1)$$

The quotient rule

To differentiate a function consisting of quotients, like these:

$$\frac{x+3}{\cos x} \qquad \frac{e^{2x}}{3\ln x} \qquad \frac{\sin x}{\cos x}$$

we can use a method called the quotient rule.

The Quotient Rule

Given a function of the form

$$y = \frac{u}{v}$$

where u and v are both functions of x,then the derivative is given by

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Or in a shorter form:

$$y' = \frac{vu' - uv'}{v^2}$$

Quotient rule — derivation

Consider the function $y = \dfrac{u(x)}{v(x)}$.

If δx is a small increase in x then there will be a corresponding increase in the values of u, v and y.

$$y + \delta y = \frac{u + \delta u}{v + \delta v}$$

Subtracting y from both sides (remembering that $y = \dfrac{u}{v}$) we get:

$$\delta y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v}$$

Combining the terms on the RHS:

$$\begin{aligned} \delta y &= \frac{(u + \delta u)v - u(v + \delta v)}{(v + \delta v)v} \\ &= \frac{uv + v\delta u - uv - u\delta v}{v^2 + v\delta v} \\ \delta y &= \frac{v\delta u - u\delta v}{v^2 + v\delta v} \end{aligned}$$

Dividing both sides by δx

$$\begin{aligned} \frac{\delta y}{\delta x} &= \frac{v\delta u - u\delta v}{\delta x(v^2 + v\delta v)} \\ \frac{\delta y}{\delta x} &= \frac{v\frac{\delta u}{\delta x} - u\frac{\delta u}{\delta x}}{v^2 + v\delta v} \end{aligned} \qquad (2)$$

If we let $\delta x \to 0$ then $\delta u \to 0$, $\delta v \to 0$ and the 2nd term in the denominator vanishes. These small changes become exact:

$$\lim_{\delta x \to 0} \frac{\delta v}{\delta x} = \frac{\mathrm{d}v}{\mathrm{d}x}, \quad \lim_{\delta x \to 0} \frac{\delta u}{\delta x} = \frac{\mathrm{d}u}{\mathrm{d}x}, \quad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{\mathrm{d}y}{\mathrm{d}x}$$

Equation (2) becomes

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}}$$

Using the quotient rule

Find the derivative of the function $y = \dfrac{3x+5}{x^2+1}$.

The RHS is composed of two functions. Let

$$u = 3x+5 \qquad \text{and} \qquad v = x^2+1$$

Differentiating each with respect to x.

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 3 \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 2x$$

The quotient rule says $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$, so:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(x^2+1)(3)-(3x+5)(2x)}{(x^2+1)^2}\\ &= \frac{3x^2+3-6x^2-10x}{(x^2+1)^2}\\ \therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3-3x^2-10x}{(x^2+1)^2}\end{aligned}$$

Differentiating $\tan x$

Find the derivative of $y = \tan x$ by expressing it as a quotient

We can use a trig identity to help in this case:

$$y = \frac{\sin x}{\cos x}$$

Let

$$u = \sin x \qquad \text{and} \qquad v = \cos x$$

Differentiating each with respect to x.

$$\frac{du}{dx} = \cos x \qquad \text{and} \qquad \frac{dv}{dx} = -\sin x$$

Using the quotient rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \end{aligned}$$

But $\cos^2 x + \sin^2 x = 1$ so

$$\frac{dy}{dx} = \frac{1}{\cos^2 x} = \sec^2 x$$

Incorporating the chain rule

We might also need to apply the chain rule during these problems.

Chain rule within the product rule

Differentiate $y = (1 - x^3)e^{2x}$

Let $u = 1 - x^3$ and $v = e^{2x}$.

To differentiate v we have to use the chain rule.

$$\frac{du}{dx} = -3x^2 \qquad \text{and} \qquad \frac{dv}{dx} = 2e^{2x}$$

Now continue as usual:

$$\begin{aligned} \frac{dy}{dx} &= (1 - x^3)2e^{2x} + e^{2x}(-3x^2) \\ &= 2(1 - x^3)e^{2x} - 3x^2e^{2x} \end{aligned}$$

Taking a common factor of e^{2x}

$$\frac{dy}{dx} = e^{2x}(2 - 3x^2 - 2x^3)$$

Finding a tangent equation

Find the equation of the tangent to the curve $f(x) = (x+4)\mathrm{e}^{-x}$ at the point (0,4).

Using the product rule; let

$$u = x+4 \quad \text{and} \quad v = \mathrm{e}^{-x}$$

Differentiate each of these with respect to x

$$u' = 1 \quad \text{and} \quad v' = -\mathrm{e}^{-x}$$

The first derivative

$$\begin{aligned} f'(x) &= uv' + vu' \\ &= (x+4)(-\mathrm{e}^{-x}) + (\mathrm{e}^{-x})(1) \\ &= -\mathrm{e}^{-x}(x+4) + \mathrm{e}^{-x} \\ f'(x) &= -\mathrm{e}^{-x}(x+3) \end{aligned}$$

The gradient of the tangent at $x = 0$ is $f'(0) = -3$.

So, the equation of the tangent is

$$y = -3x + c$$

At the point $(0,4)$:

$$\begin{aligned} 4 &= -3(0) + c \\ \therefore c &= 4 \end{aligned}$$

This means the tangent equation at $(0,4)$ is

$$y = 4 - 3x$$

Test yourself...

Now try to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for the following functions.

1. $y = x^3 \ln x$.
2. $y = \dfrac{6x}{x^3 + 1}$.
3. $y = 6x\sqrt{3x+8}$
4. Use quotient rule to find $\dfrac{\mathrm{d}}{\mathrm{d}x}[\cot 2x]$
   (Hint: $\cot x \equiv \cos x / \sin x$)

Answers:

1. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^2(3\ln x + 1)$.
2. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{6(1-2x^3)}{(x^3+1)^2}$.
3. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3(9x+16)}{\sqrt{3x+8}}$.
4. $-2\mathrm{cosec}^2 2x$