Chain Rule

Calculus 2

INU0115/515 (Maths 2)

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Composing and decomposing functions

A composite function is a function within some other function. For example, if

$$f(x) = x^{10} \quad \text{and} \quad g(x) = 4x + 3$$

we can make composite functions like this:

$$fg(x) = (4x+3)^{10} \qquad \text{or} \qquad gf(x) = 4x^{10} + 3$$

To find a derivative, it is useful to do the reverse process; to split a composite function into simpler functions.

For example, given the function

$$h(x) = \sqrt{2x+5}$$

We can split this into two functions $f(x) = \sqrt{x}$ and $g(x) = 2x + 5$ (so that $h(x) = fg(x)$).

Although we'll be changing the labels (using y and u rather than f and g) we will be doing the exact same thing on the next slides.

The Chain Rule

Composite functions are differentiated with the *chain rule*.

The Chain Rule

Given the composite function $y = y(u)$, where $u = u(x)$ then the gradient is found by:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

The chain rule can be extended to more complex situations by adding 'links' to the formula. Given the composite function $y = y(v)$, where $v = v(u)$ and $u = u(x)$ then the gradient is found by:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

and so on.

The proof of the chain rule is just beyond the scope of this course, but if you're interested then more details can be seen at `http://en.wikipedia.org/wiki/Chain_rule`.

Using the chain rule

Find the derivative of the curve given by $y=(x^2+3)^7$.

This is a composite function. We decompose it into simpler functions:

$$y=u^7 \quad \text{where} \quad u=x^2+3$$

The corresponding derivatives of these are:

$$\frac{\mathrm{d}y}{\mathrm{d}u}=7u^6 \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}x}=2x$$

The chain rule requires multiplyinging these derivatives together:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 7u^6\times 2x \\ &= 14xu^6\end{aligned}$$

Change back to the original variable. Substitute $u=x^2+3$ into the previous answer to get:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=14x(x^2+3)^6$$

Using the chain rule

Find the gradient of the curve $y = \cos 5x$.

Decompose into simpler functions:

$$y = \cos u \qquad \text{and} \qquad u = 5x$$

Differentiating these we get:

$$\frac{\mathrm{d}y}{\mathrm{d}u} = -\sin u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 5$$

Applying the chain rule gives:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\sin u \times 5\\ &= -5\sin u\\ \therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= -5\sin 5x\end{aligned}$$

Using the chain rule

Find the gradient of the curve $y = \mathrm{e}^{x^2}$ at the point $x = \frac{1}{2}$.

Decompose into simpler functions:

$$y = \mathrm{e}^u \quad \text{and} \quad u = x^2$$

Differentiating these we get:

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \mathrm{e}^u \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$

Applying the chain rule gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x\mathrm{e}^u = 2x\mathrm{e}^{x^2}$$

At point where $x = \frac{1}{2}$ the gradient is $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{\frac{1}{4}}$

Chain rule ‘by inspection’

The chain rule gives an answer after multiplying the derivatives of two functions together. We’ll call these the *inner* and *outer* functions.

y	Outer	Inner $(*)$	Chain Rule	y'
$\sin 3x$	$\sin(*)$	$3x$	$3 \times \cos(*)$	$3\cos 3x$
e^{x^2}	$e^{(*)}$	x^2	$2x \times e^{(*)}$	$2xe^{x^2}$
$(6-7x)^{10}$	$(*)^{10}$	$6-7x$	$(-7)\times 10(*)^9$	$-70(6-7x)^9$

The chain rule can often be applied without needing to write down all of the steps shown previously. If you can identify the inner and outer functions then it is easy to differentiate and multiply them.

Chain rule (by inspection)

Find the derivative of $y = \tan(x^2 - \pi)$

The derivative of ‘tan’ is ‘$\sec^2$’, so the one of the derivatives is $\sec^2(x^2-\pi)$.

The derivative of $x^2 - \pi$ is just $2x$ (since π is constant). Therefore:

$$\frac{dy}{dx} = \sec^2(x^2-\pi) \times 2x = 2x\sec^2(x^2-\pi)$$

Chain rule by inspection

Find the gradient function of

$$y = \ln(x^3+1) + 10\sin\left(\tfrac{1}{2}x^4\right)$$

Each term is differentiated separately. The derivative of $\ln(*)$ will contain $\frac{1}{*}$ multiplied by $(*)'$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(x^3+1)\right] = \frac{1}{x^3+1} \times 3x^2 = \frac{3x^2}{x^3+1}$$

Now the second term. The derivative of $\sin(*)$ will contain $\cos(*)$ multipled by $(*)'$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[10\sin\left(\tfrac{1}{2}x^4\right)\right] = 10\cos\left(\tfrac{1}{2}x^4\right) \times 2x^3 = 20x^3\cos\left(\tfrac{1}{2}x^4\right)$$

Therefore the gradient function is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{x^3+1} + 20x^3\cos\left(\tfrac{1}{2}x^4\right)$$

Chain rule: more links in the chain

Given the function

$$y = \sqrt[3]{\sin(x^4)}$$

Find the gradient function $\frac{dy}{dx}$.

We start with the innermost function; let $u = x^4$.

Rewrite the original function as

$$y = \sqrt[3]{\sin u}$$

Let $v = \sin u$ so that

$$y = \sqrt[3]{v} = v^{\frac{1}{3}}$$

This function can be differentiated without further substitutions.

$$\frac{dy}{dv} = \frac{1}{3} v^{-\frac{2}{3}}$$

We also differentiate the functions v and u:

$$\frac{dv}{du} = \cos u \quad , \frac{du}{dx} = 4x^3$$

The gradient is assembled from the derivatives like this:

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{du}\frac{du}{dx}$$

(See how the terms on the RHS appear to 'cancel' to give $\frac{dy}{dx}$?).

$$\frac{dy}{dx} = \left(\frac{1}{3} v^{-\frac{2}{3}}\right)(\cos u)(4x^3)$$

Change the variables back to x:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{3}(\sin x^4)^{-\frac{2}{3}}(4x^3)\cos(x^4) \\ &= \frac{4x^4 \cos x^4}{3\sqrt[3]{\sin^2 x^4}}\end{aligned}$$

Test yourself...

Use your knowledge of differentiation answer the following questions.

1. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \sec 4x$.
2. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \sqrt{4x+9}$.
3. Find $\frac{\mathrm{d}}{\mathrm{d}x}(\ln 2x)$
4. Find the tangent equation to the curve $f(x) = (1-x^2)^3$ at the point where $x = \frac{1}{2}$.

Answers:

1. $\frac{\mathrm{d}y}{\mathrm{d}x} = 4\sec 4x \tan 4x$.
2. $\frac{\mathrm{d}y}{\mathrm{d}x} = 2(4x+9)^{-\frac{1}{2}}$.
3. $\frac{1}{x}$
4. $y = -\frac{27}{16}x + \frac{81}{16}$