STANDARD DERIVATIVES

CALCULUS 1

INU0115/515 (MATHS 2)

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Notation

Calculus was developed by many mathematicians working independently in different countries. As a result, there are several different ways of expressing the same ideas!For example, it can be shown that if:

$$y = x^2 \qquad \text{then} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 2x$$

But function notation varies so we can express the same relationship like this:

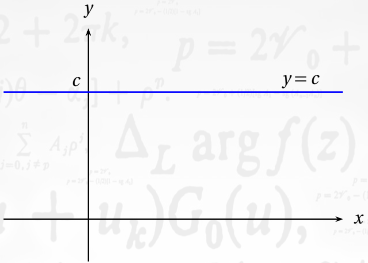
$$f(x) = x^2 \qquad \text{then} \quad f'(x) = 2x$$

Here the prime ($'$) symbol represents differentiation of the function.

Whenever we differentiate x^2 we get $2x$ - it doesn't matter what the function is called. The process of differentiation is sometimes denoted by the symbol $\frac{\mathrm{d}}{\mathrm{d}x}$ (although it doesn't have to be x), so that we could write:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x$$

Derivative of a constant



Consider the equation $y = c$ where c is a constant.

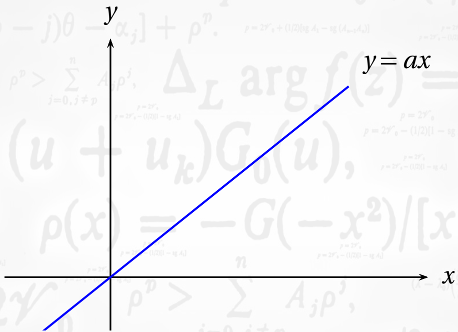
This equation represents a straight line parallel to the x-axis. The gradient of this line is zero - the value of y doesn't change as the value of x changes.

Since the gradient is zero the value of $\frac{dy}{dx}$ will also be zero. Therefore, when we differentiate a constant:

$$\boxed{\frac{d}{dx}(c) = 0}$$

Derivative of a linear term

Consider the equation $y = ax$ where a is a constant. This equation represents a straight line passing through the origin with gradient a.



It's easy to show from 'first principles' that $\frac{dy}{dx} = a$. So:

$$\boxed{\frac{d}{dx}(ax) = a}$$

Looking for patterns

We can apply the formula for 'first principles' to some powers of x to get the following results:

y	x	x^2	x^3	x^{-1}	x^{-2}
$\frac{dy}{dx}$	1	$2x$	$3x^2$	$-x^{-2}$	$-2x^{-3}$

Without a going into a rigorous proof, there appears to be a pattern for differentiating expressions of the form x^n:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Although we only considered integer values of n, the rule works for any value of n.

In words we might remember this rule as 'subtract one from the power and multiply by the old power'

We'll look at some examples of this rule now.

Differentiating negative powers

Differentiate the expression $y = -x^{-4}$.

Differentiate each term separately:

$$\frac{dy}{dx} = -(-4)x^{-4-1} = 4x^{-5}$$

Differentiating fractional powers

Find the derivative of $y = \sqrt{x}$.

First, we write the expression using index notation; $y = x^{\frac{1}{2}}$. The derivative is

$$\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$$

Using the rules of indices, it can also be given in the form $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$.

Linearity rules

Differentiation is *distributive* over addition and subtraction[1]. This simply means that if we have an expression whose terms are separated by addition or subtraction then we can find the derivative by differentiating term by term.

Differentiating term by term

Find the derivative of the expression $x^5 - \dfrac{1}{x^3}$.

First, let's write it as: $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^5 - x^{-3})$

Now differentiate term by term:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^5 - x^{-3}) = 5x^4 - (-3)x^{-4} = 5x^4 + 3x^{-4}$$

Alternatively, we can change back from negative powers to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^5 - \frac{1}{x^3}\right) = 5x^4 + \frac{3}{x^4}$$

[1] It is not distributive over multiplication and division; we will use the product and quotient rule in a later section.

The expression ax^n

Using the ‘first principles’ method we can find these derivatives.

y	$\frac{dy}{dx}$	
x^2	$2x$	
$2x^2$	$4x$	$2(2x)$
$3x^2$	$6x$	$3(2x)$
$4x^2$	$8x$	$4(2x)$

y	$\frac{dy}{dx}$	
x^3	$3x^2$	
$2x^3$	$6x^2$	$2(3x^2)$
$3x^3$	$9x^2$	$3(3x^2)$
$4x^3$	$12x^2$	$4(3x^2)$

The coefficient does not affect the differentiation; we use the same rule as before to change the power.

It can be shown that any number a multiplying the original expression also multiplies the derivative. The rule is:

$$\frac{d}{dx}(ax^n) = a\frac{d}{dx}(x^n) = anx^{n-1}$$

Differentiating powers

Find the gradient of the function

$$y = 4x^5 - 3x^2 - 10\sqrt{x} + 5 \ , \ x \geq 0$$

at the point where $x = 1$.

First, we write the function using index notation:

$$y = 4x^5 - 3x^2 - 10x^{\frac{1}{2}} + 5$$

The derivative is obtained by differentiating term by term on the RHS:

$$\begin{aligned} \frac{dy}{dx} &= 4(5x^4) - 3(2x) - 10(\tfrac{1}{2})x^{-\frac{1}{2}} \\ &= 20x^4 - 6x - 5x^{-\frac{1}{2}} \end{aligned}$$

If we wish, we can change back to having square-roots in the final answer:

$$\frac{dy}{dx} = 20x^4 - 6x - \frac{5}{\sqrt{x}}$$

Substitute the value $x = 1$ into this to get $\frac{dy}{dx} = 9$.

Higher derivatives

The process of differentiation can be applied more than once if necessary. For example, consider the case of the falling ball whose distance y from the initial height described by the function

$$y = 5t^2$$

If we differentiate this with respect to t, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 10t$$

This expression is the sometimes called the *first derivative* and it describes how distance changes with time (it is the speed of the ball).

We can differentiate this again to find out how the speed is changing with time - the acceleration. The *second derivative* is written like this:

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 10$$

The notation contains indices but they are not powers; they only indicate how many times the expression has been differentiated.

Higher derivatives

Given the function $y = x^5 - 4x^2 + 12$ find value of the third derivative at $x = \frac{1}{2}$.

It's easy to differentiate this function! The first derivative:

$$\frac{dy}{dx} = 5x^4 - 8x$$

The second derivative:

$$\frac{d^2y}{dx^2} = 20x^3 - 8$$

And the third derivative:

$$\frac{d^3y}{dx^3} = 60x^2$$

At the point where $x = \frac{1}{2}$, we have $\frac{d^3y}{dx^3} = 60 \times \left(\frac{1}{2}\right)^2 = 15.$

Standard derivatives

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\ln x$	$\frac{1}{x}$
e^x	e^x
a^x	$(\ln a)a^x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$

The derivatives on this page can, in theory, be derived by the first principles method.

(It may be difficult to prove some of them)

However, in all work from now, you can quote the derivatives of these functions directly without giving a proof.

Some examples to follow on the next couple of slides.

Using standard derivatives

Differentiate the function $y = x^2 + \sin x + \ln x$ to find the first and second derivatives.

Differentiate each term separately:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + \cos x + \frac{1}{x},$$

Differentiate the terms again to get the second derivative:

$$\begin{aligned} \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= 2 + (-\sin x) - x^{-2} \\ &= 2 - \sin x - \frac{1}{x^2} \end{aligned}$$

Finding a tangent equation

Consider the curve described by $f(x) = 3\cos x$. Find the equation of the tangent to the curve at the point where $x = \frac{3\pi}{2}$ rads.

Differentiate to get the gradient function

$$f'(x) = -3\sin x$$

The value of this at $x = \frac{3\pi}{2}$ is $-3\sin(\frac{3\pi}{2}) = -3(-1) = 3$

The tangent line has a gradient of 3. The equation of the tangent is:

$$y = 3x + c$$

At the point $x = \frac{3\pi}{2}$, we calculate that $y = 3\cos(\frac{3\pi}{2}) = 0$. Put these into the tangent equation:

$$0 = 3(\frac{3\pi}{2}) + c \qquad \Rightarrow c = -\frac{9\pi}{2}$$

The tangent equation is $y = 3x - \frac{9\pi}{2}$.

Finding a normal equation

A curve has the equation $y = \sqrt{x}(k - x)$, where k is a constant.
Given that the gradient of the curve is $\sqrt{2}$ at the point P where $x = 2$,

1. find the value of k
2. show that the normal to the curve at P has the equation

$$x + \sqrt{2}y = c$$

   where c is an integer to be found.

Part (1). The curve can be written as $y = kx^{\frac{1}{2}} - x^{\frac{3}{2}}$. Differentiate:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}kx^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} = \frac{k}{2\sqrt{x}} - \frac{3}{2}\sqrt{x}$$

At $x = 2$, we know $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{2}$, so $\frac{k}{2\sqrt{2}} - \frac{3}{2}\sqrt{2} = \sqrt{2}$

Multiply both sides by $\sqrt{2}$ to rationalise then rearrange for k:

$$\frac{k}{2} - \frac{3}{2}(2) = 2 \quad \Rightarrow k = 10$$

Therefore the curve equation is $y = \sqrt{x}(10 - x)$.

Finding a normal equation

A curve has the equation $y = \sqrt{x}(k - x)$, where k is a constant.
Given that the gradient of the curve is $\sqrt{2}$ at the point P where $x = 2$,

1. find the value of k
2. show that the normal to the curve at P has the equation

$$x + \sqrt{2}y = c$$

where c is an integer to be found.

Part (2). Given that the tangent gradient is $\sqrt{2}$ then the normal gradient must be $-\frac{1}{\sqrt{2}}$ (using $m_1 m_2 = -1$). The equation of the normal must have the form

$$y = -\frac{1}{\sqrt{2}}x + c$$

On the curve, the point $x = 2$ corresponds to $y = \sqrt{2}(10 - 2) = 8\sqrt{2}$. Putting these into the straight line equation gives:

$$8\sqrt{2} = c - \frac{1}{\sqrt{2}}2 + c \qquad \Rightarrow c = 8\sqrt{2} + \frac{2}{\sqrt{2}} = 9\sqrt{2}$$

So the normal equation is $y = -\frac{1}{\sqrt{2}}x + 9\sqrt{2}$. Multiply both sides by $\sqrt{2}$ and rearrange to the required form:

$$\sqrt{2}y = -x + 18 \qquad \Rightarrow x + \sqrt{2}y = 18$$

Test yourself...

Use your knowledge of differentiation answer the following questions.

1. Find $\dfrac{dy}{dx}$ when $y = x^7 - \frac{1}{3}x^3 + 2$.
2. Find $\dfrac{dy}{dx}$ when $y = x^2(x^{\frac{1}{2}} + 1)$.
3. Find $\dfrac{d}{dx}\left(3\cos x - \dfrac{\sin x}{3} + e^x\right)$
4. Find the tangent equation to the curve $f(x) = x^3 - x + 3$ at the point where $x = -1$.

Answers:

1. $\dfrac{dy}{dx} = 7x^6 - x^2$.
2. $\dfrac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}} + 2x$.
3. $-3\sin x - \dfrac{\cos x}{3} + e^x$
4. $y = 2x + 5$