Differentiation

Calculus 1

INU0115/515 (Maths 2)

Dr Adrian Jannetta MIMA CMath FRAS

Introduction

The foundations of calculus were laid in the 17th century during a time when scientists were trying to understand and solve important problems in physics and astronomy.

Differential calculus (or differentiation) is about describing how functions (or their graphs) are changing. For example:

- The function which describes how distance changes with time is called velocity.
- The function which describes how velocity changes with time is called acceleration.

Integral calculus (or integration) was developed to solve problems requiring the calculation of areas or volumes. Several early mathematicians (Eudoxus, Archimedes, Liu Hui, Cavalieri and others) introduced ideas which led to the development of integration.

Differentiation and integration are connected via the Fundamental Theorem of Calculus.[1].

[1] More about this in semester 2!

Founders of Calculus

Isaac Newton (1642 — 1727) and Gottfried Wilhelm von Leibniz (1646 — 1716). The two mathematicians invented and developed calculus independently of each other.

We'll begin our study of calculus by focussing on differentiation — finding how functions change. At the heart of this process is the notion of *limits*.

Definition of a derivative

In these notes we'll try to understand the concept of a derivative in two different ways.

Physical in this interpretation we'll examine the familiar concepts of position and velocity to understand what we mean by 'rate of change'. We will examine the simple case of an accelerating car and attempt to calculate its speed at a given time.

Geometric in this interpretation we'll see that for a curve plotted on a graph we can specify a rate of change - the gradient. You should recall that the gradient m of a straight line is defined as

$$m = \frac{\text{Change in the value of } y}{\text{Change in the value of } x} = \frac{\delta y}{\delta x}$$

Throughout this section the symbols δ (delta) or Δ (Delta) are used to denote small changes in the given variable.

Physical interpretation

Here is a picture of a car — can you tell if it's moving or stationary?

Without anything to compare it to, it's not possible to tell.

If we knew the position just before or just after this picture was taken then we'd be able to tell.

1 second earlier

1 second later

Then perhaps we could find the speed of the car by using the formula for average speed:

$$\text{Average speed} = \frac{\text{Distance moved}}{\text{Time taken}}$$

Average speed

Suppose a car is accelerating from rest so that the distance y meters, that it has travelled after t seconds is given by the equation:

$$y = t^2$$

The following sequence shows the position of the car at intervals of 1 second.

So after 1 second the car has travelled $y = 1^2 = 1$ m.

After 2 seconds the car has travelled $y = 2^2 = 4$ m.

After 3 seconds the car has travelled $y = 3^2 = 9$ m.

After 4 seconds the car has travelled $y = 4^2 = 16$ m.

...and so on. It is easy to calculate the distance. We can also find the average speed.

For example the car in the picture travels 16 metres in 4 seconds. So the average speed is $4\,\mathrm{m\,s^{-1}}$.

But suppose we want to know the *instantaneous speed* of the car after 4 seconds? How fast is the car actually moving after 4 seconds?

Average vs instantaneous speed

We can do a numerical experiment to estimate the instantaneous speed of the car after 4 seconds.

Between 4 and 5 seconds the car moves from 16 m to 25 m from the start. This means that the average speed was:

$$\text{Average speed} = \frac{25-16}{5-4} = 9\,\text{m}\,\text{s}^{-1}$$

Since the car was accelerating during that time interval then the average speed is going to be different to the actual (instantaneous) speed at $t = 4$s.

We can get a better estimate by observing the motion of a smaller time interval. The speed of the car does not have so much time to change and the average speed should be closer in value to the instantaneous speed.

Average vs instantaneous speed

We'll calculate the average speed between $t = 4$s and $t = 4.1$s.

After 4 seconds the car has moved 16 m.

After 4.1 seconds the car has moved $4.1^2 = 16.81$ m.

Therefore:

$$\text{Average speed} = \frac{16.81 - 16}{4.1 - 4} = \frac{0.81}{0.1} = 8.1 \ \text{m s}^{-1}$$

We might expect this to be close in value to the actual speed at $t = 4$s, but we can use a smaller time interval to be more certain.

Average vs instantaneous speed

We can improve our accurate estimate of the speed at 4 seconds by using an even smaller time interval.

Calculate the average speed between $t = 4$s and $t = 4.01$s.

After 4 seconds the car has moved 16m.

After 4.01 seconds the car has moved $4.01^2 = 16.0801$m.

Therefore:

$$\text{Average speed} = \frac{16.0801 - 16}{4.01 - 4} = \frac{0.0801}{0.01} = 8.01\,\text{m}\,\text{s}^{-1}$$

This must be much closer to the instantaneous speed at $t = 4$s than the previous estimate. Can you see a pattern emerging from these calculations?

Average vs instantaneous speed

We can repeat the calculation using a smaller time interval Δt each time.

Here's a table showing the results of those calculations.

Time interval Δt	Average speed
0.1 s	$8.1\,\mathrm{m\,s^{-1}}$
0.01 s	$8.01\,\mathrm{m\,s^{-1}}$
0.001 s	$8.001\,\mathrm{m\,s^{-1}}$
0.0001 s	$8.0001\,\mathrm{m\,s^{-1}}$
0.00001 s	$8.00001\,\mathrm{m\,s^{-1}}$

The calculated speeds are approaching $8\,\mathrm{m\,s^{-1}}$ more closely.

We cannot make the time interval zero because we are not allowed to divide by zero.

All we can say is that the average speed approaches a value of $8\,\mathrm{m\,s^{-1}}$ as the time interval approaches zero. This is the instantaneous speed of the car.

Definition of instantaneous speed

So far we have done some calculations which show that the average speed approaches a limiting value, called the instantaneous speed (of $8\,\mathrm{m\,s^{-1}}$) as the time interval approaches zero.

Mathematically, we can write this statement as:

$$\lim_{\delta t \to 0} \frac{\delta y}{\delta t} = 8$$

The quantity on the LHS of this statement represents the instantaneous speed. Mathematicians use a shorter notation to represent it:

$$\lim_{\delta t \to 0} \frac{\delta y}{\delta t} \equiv \frac{\mathrm{d}y}{\mathrm{d}t}$$

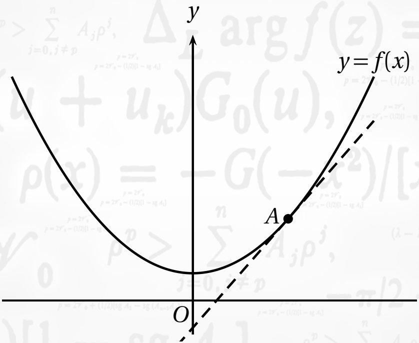
So in this case we found that the speed, $\frac{\mathrm{d}y}{\mathrm{d}t} = 8$ at $t = 4$s.

The quantity $\frac{\mathrm{d}y}{\mathrm{d}t}$ is also called the *derivative* of y with respect to t.

Geometric interpretation

Gradient of a curve

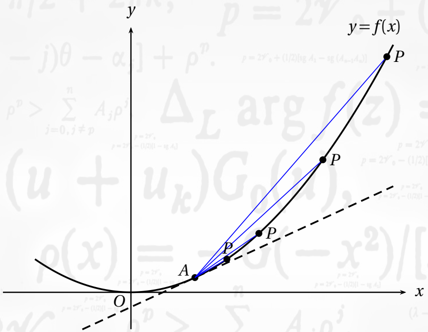
The gradient (or *slope*) of a curve at a given point has the same value as the gradient of a tangent line touching the curve at that point.



The tangent line to a point A on the curve has the same slope as the curve itself at point A.

Gradient of the curve $y = x^2$

Here is the graph of $y = x^2$.

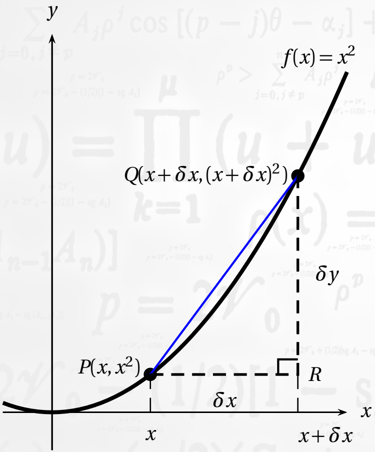


We are trying to find the gradient of the curve at A.

The gradient of the chord line AP becomes closer in value to the gradient of the curve at A (shown by the dashed tangent line) if we slide P closer to A.

Finding the gradient from first principles

Let's find the gradient of $y = x^2$ at the point P. We then consider a point Q a little further away on the curve.



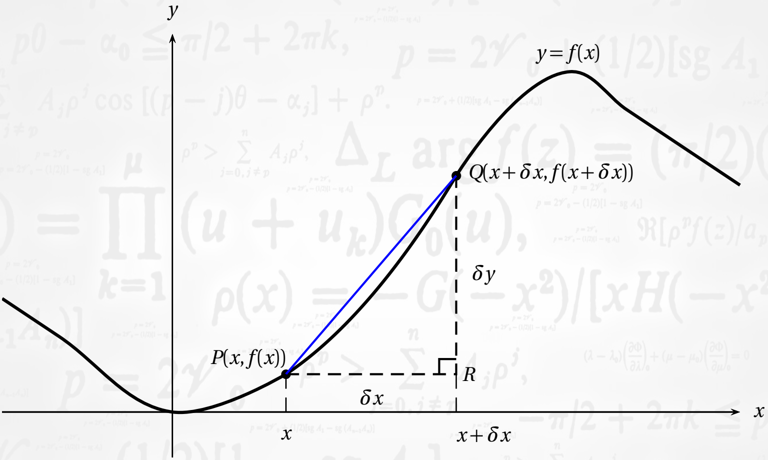
Gradient of PQ:

$$\frac{\delta y}{\delta x} = \frac{(x+\delta x)^2 - x^2}{\delta x}$$
$$= \frac{x^2 + 2x\delta x + (\delta x)^2 - x^2}{\delta x}$$
$$= \frac{2x\delta x + (\delta x)^2}{\delta x}$$
$$\frac{\delta y}{\delta x} = 2x + \delta x$$

We find the gradient at P by letting Q slide down the curve (so that $\delta x \to 0$):

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{\mathrm{d}y}{\mathrm{d}x} = 2x$$

The general case: $y = f(x)$



$$\text{Gradient at } P \quad = \quad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Differentiation from first principles

The method for obtaining the derivative shown on the previous slides is called the first principles method. The process of finding the derivative of a function is usually called *differentiation*.

First principles formula

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

If we use the derivative formula then we are said to be doing *differentiation from first principles*.

On the next slide you will see how the formula can be used to differentiate some simple functions.

First principles: examples

Differentiating from first principles

Find the derivative of $y = 2x^3$ from first principles.

Using the first principles formula:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{\delta x \to 0} \frac{2(x+\delta x)^3 - 2x^3}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{2(x^3 + 3x(\delta x)^2 + 3x^2\delta x + (\delta x)^3) - 2x^3}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{2x^3 + 6x(\delta x)^2 + 6x^2\delta x + 2(\delta x)^3 - 2x^3}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{6x(\delta x)^2 + 6x^2\delta x + 2(\delta x)^3}{\delta x} \\
&= \lim_{\delta x \to 0} 6x\delta x + 6x^2 + 2(\delta x)^2
\end{aligned}$$

If we let $\delta x \to 0$ this expression becomes $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 6x^2$.

First principles: examples

Differentiating from first principles

Find the derivative of $y = \frac{5}{x}$ from first principles.

Using the first principles formula:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{\delta x \to 0} \frac{\frac{5}{x+\delta x} - \frac{5}{x}}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{\frac{5x-5(x+\delta x)}{(x+\delta x)x}}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{5x - 5x - 5\delta x}{\delta x(x+\delta x)x} \\
&= \lim_{\delta x \to 0} \frac{-5\delta x}{\delta x(x+\delta x)x} \\
&= \lim_{\delta x \to 0} \frac{-5}{(x+\delta x)x}
\end{aligned}$$

If we let $\delta x \to 0$ this expression becomes $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{5}{x^2}$.

A slightly different approach...

Consider the function

$$y = f(x)$$

A small change δx in the value of x creates a small change δy in the value of y so that:

$$y + \delta y = f(x + \delta x)$$

Rearrange to get

$$\delta y = f(x + \delta x) - y$$

But since $y = f(x)$ we'd write this as:

$$\delta y = f(x + \delta x) - f(x)$$

To find the gradient, first, divide both sides by δx

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

The gradient is the value of this function as δx approaches zero:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

Differentiating from first principles

Use the method of first principles to find the gradient of $y = 10x - 2x^2$.

A small change in x creates a corresponding change in the value of y:

$$y + \delta y = 10(x + \delta x) - 2(x + \delta x)^2$$

Rearrange to get

$$\begin{aligned} \delta y &= 10(x + \delta x) - 2(x + \delta x)^2 - y \\ &= 10(x + \delta x) - 2(x + \delta x)^2 - (10x - 2x^2) \end{aligned}$$

Expand the brackets and simplify:

$$\begin{aligned} \delta y &= 10x + 10\delta x - 2(x^2 + 2x\delta x + (\delta x)^2) - 10x + 2x^2 \\ &= 10x + 10\delta x - 2x^2 - 4x\delta x - 2(\delta x)^2 - 10x + 2x^2 \\ &= 10\delta x - 4x\delta x - 2(\delta x)^2 \end{aligned}$$

Divide through by δx:

$$\frac{\delta y}{\delta x} = \frac{10\delta x - 4x\delta x - 2(\delta x)^2}{\delta x} = 10 - 4x - 2\delta x$$

Finally, let $\delta x \to 0$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 10 - 4x$$