Modulus equations and inequalities

Algebra 5

INU0114/514 (Maths 1)

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Objectives

In this presentation we'll see a few different ways of solving equations and inequalities involving the modulus function.

You should already know:

- The piecewise definition of the modulus function $f(x) = |x|$.
- How to sketch straight line graphs $y = mx + c$.
- How to sketch graphs of $y = \left|f(x)\right|$. In this context, $f(x)$ will be a linear function.
- Know how to factorise or solve quadratics.

The method we'll use to solve equations and inequalities will be either:

- Using algebra, or
- Using a graph.

Equations and inequalities with a modulus

In this presentation we'll see how solve equations such as

$$|4x-10|=|3x|$$

and inequalities like

$$|x+10|\leq 9x$$

To do this we'll need to recall the definition of the modulus function given by

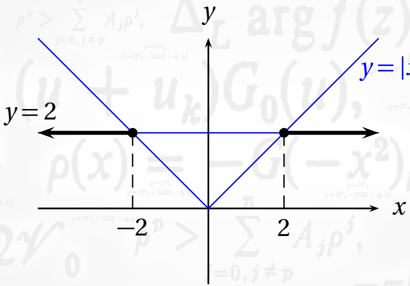
$$f(x)=|x|=\begin{cases} x & x\geq 0 \\ -x & x<0 \end{cases}$$

There several possible ways to solve equations and inequalities: using algebra or using a graph. In the end, it is up to you to choose the best way to solve problems.

Method #1: Graphical

Consider the inequality $|x| \geq 2$.

On a graph we plot the functions $y = |x|$ and $y = 2$



The solutions are $x \leq -2$ and $x \geq 2$.

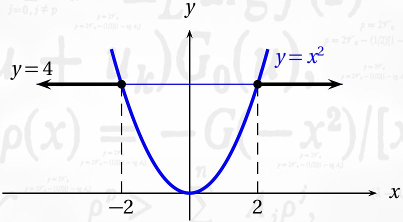
Method #2: Squaring both sides

The previous discussion showed that the inequality $|x| \geq 2$ had solutions $x \leq -2$ and $x \geq 2$.

The same solutions also satisfy

$$x^2 \geq 4$$

We plot the graphs $y = x^2$ and $y = 4$:



Inspect the graph to where the curve $y = x^2$ has larger values than $y = 4$.

The solutions are $x \leq -2$ and $x \geq 2$ (as expected!).

Although this method is simple to apply — we always need to check for extraneous (false) solutions. Always verify the solutions with the original equation or inequality.

Method #3: Using the modulus definition

Recall that

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

For example: to solve

$$|2x + 1| = 5$$

We instead solve the following two equations:

$$2x + 1 = 5 \qquad \text{and} \qquad -(2x+1) = 5$$

The first equation gives $x = 2$. Solving the second equation gives $x = -3$.

Equations with modulus functions

Solve the equation

$$|x+2| = |x-1|$$

(Note: we can't answer this by just removing the modulus!)

Solve, for example, by squaring both sides:

$$(x+2)^2 = (x-1)^2$$

Expand the brackets and solve:

$$\begin{aligned} x^2+4x+4 &= x^2-2x+1 \\ 4x+4 &= -2x+1 \\ 6x &= -3 \\ x &= -\tfrac{1}{2} \end{aligned}$$

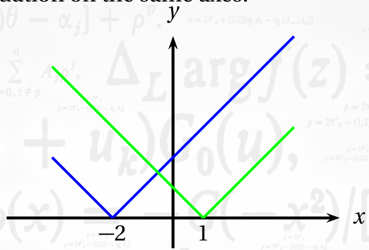
The only solution is $x = -\frac{1}{2}$ (it's easy to verify this in the original equation).

Equation with a modulus (graphical method)

Solve the equation

$$|x+2| = |x-1|$$

Plot each side of the equation on the same axes:



The intersection is found by solving

$$\begin{aligned} x+2 &= -(x-1) \\ x+2 &= -x+1 \\ 2x &= -1 \end{aligned}$$

Therefore $x = -\frac{1}{2}$.

Equation with a modulus function

Solve the inequality

$$|2x+3| < 5$$

This equation actually represents two equations which can be solved separately.

$$2x+3 < 5 \quad \text{and} \quad -(2x+3) < 5$$

Solve the first to get $x < 1$.

The second is solved like this:

$$\begin{aligned} -2x-3 &< 5 \\ -2x &< 8 \\ 2x &> -8 \\ x &> -4 \end{aligned}$$

The two solutions are $x > -4$ and $x < 1$.

Combine these into a single inequality: $-4 < x < 1$.

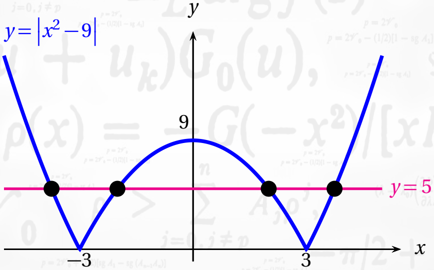
Number of solutions

Consider the equation

$$\left|x^2 - 9\right| = 5$$

How many solutions should we expect to find? (Don't find x values first!)

A graph gives the solution pretty quickly; the solutions are the intersections between $y = \left|x^2 - 9\right|$ and the line $y = 5$.



The graph shows we should expect four solutions to this question.

(If you solve it: $x = -\sqrt{14}, -2, 2$ and $\sqrt{14}$).