Modulus Functions

Algebra 5

INU0114/514 (Maths 1)

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Objectives

In this session we will do the following:

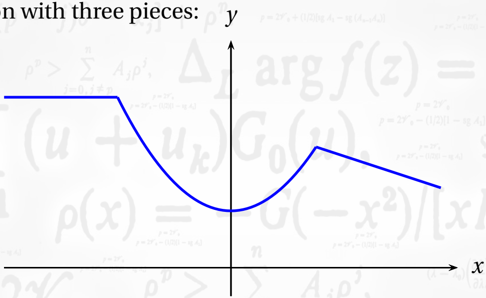
- Defining piecewise functions
- The modulus function
- Graphs of modulus functions
- Solving equations containing the modulus function

It is assumed that you can already sketch graphs for linear and quadratic functions.

Piecewise function definitions

Functions can be defined in pieces over a given domain and then stitched together to make a new function.

Here is a function with three pieces:



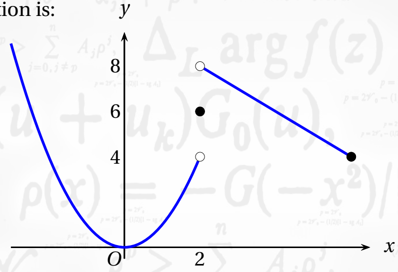
To define this function over the entire curve we need a piecewise definition.

Piecewise function definitions are useful for defining relationships where the behaviour of the variables changes dramatically at certain places.

Let's define a function with the following properties:

- It is x^2 when x is less than 2
- It has the value 6 when x is exactly 2
- It is $10-x$ when x is more than 2 and less than or equal to 6.

The graph of the function is:



We define a piecewise function like this:

$$f(x) = \begin{cases} x^2 & x < 2 \\ 6 & x = 2 \\ 10 - x & 2 < x \leq 6 \end{cases}$$

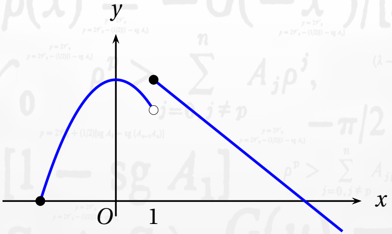
A piecewise function

Sketch the graph of the following function:

$$f(x) = \begin{cases} 4 - x^2 & -2 \le x < 1 \\ 5 - x & x \ge 1 \end{cases}$$

The graph is a part of an n-shaped quadratic from $-2 \le x < 0$. We used closed and open circles to mark the ends of the curve.

The graph is a straight-line for positive values of x. We need a closed circle to represent the start of the line.



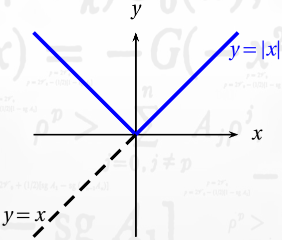
The modulus function

The modulus function $f(x) = |x|$, which is also called the *absolute value function*, is defined so that

- When x is negative it is $-x$
- When x is zero or positive it is x

Or using piecewise notation as: $f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

And the graph:



The modulus function has the effect of reflecting those parts of $y = x$ below the x-axis in the x-axis.

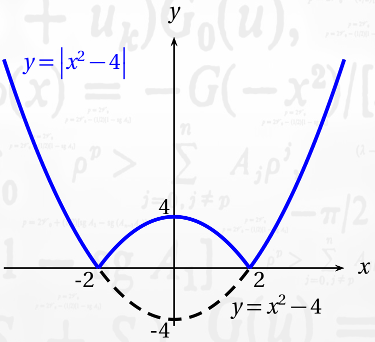
The graph of $\left|f(x)\right|$

The modulus of a function

Sketch the graph of the function $f(x) = \left|x^2 - 4\right|$

The graph can be obtained easily by first sketching $y = x^2 - 4$.

Modify the shape by reflecting of those parts below the x-axis above it.

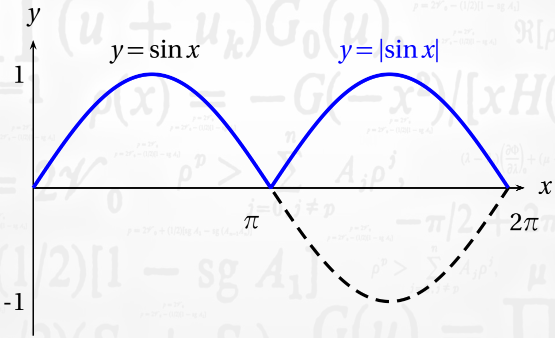


The modulus of a function

Sketch the graph of the function $y = |\sin x|$, $0 \leq x \leq 2\pi$

First, sketch $y = \sin x$.

Modify the shape by reflecting of those parts below the x-axis above it.



The graph of $f(|x|)$

If only part of the function contains a modulus then we consider those parts separately.

Remember: the modulus function has a piecewise definition that we can use.

Function of a modulus

Sketch the graph of $f(x) = 2|x| - 3$.

In this problem, we'll split the function into two pieces to understand.

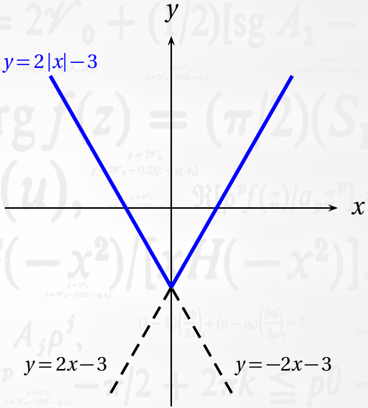
The piecewise definition is:

$$f(x) = \begin{cases} 2x - 3 & x \geq 0 \\ -2x - 3 & x < 0 \end{cases}$$

We know how to sketch straight line graphs, so we can do this in two pieces.

Here is how to sketch the graph of $f(x) = 2|x| - 3$.

- Sketch the axes...
- Sketch the line $y = 2x - 3$.
- ...and the line $y = -2x - 3$.
- Highlight $y = 2x - 3$ to the right of the y-axis...
- Highlight $y = -2x - 3$ to the left of the y-axis.



Function of a modulus

Sketch the graph of $f(x) = |x| + |x-3|$.

Let's consider various values of x and how they affect $f(x)$.

When $x > 3$ then we don't need modulus bars: simply

$$f(x) = x + x - 3 = 2x - 3 \text{, for } x > 3$$

When $0 \le x \le 3$ then we only need to consider the modulus bars on the second term:

$$f(x) = x - (x-3) = 3 \text{, for } 0 < x < 3$$

When $x < 0$ then the function can be written

$$f(x) = -x - (x-3) = -2x + 3 \text{, for } 0 < x < 3$$

The piecewise definition is:

$$f(x) = \begin{cases} -2x+3 & x < 0 \\ 3 & 0 \le x \le 3 \\ 2x-3 & x > 3 \end{cases}$$

We know how to sketch straight line graphs, so we can do this in two pieces.

We saw that the function $f(x) = |x| + |x-3|$ is equivalent to

$$f(x) = \begin{cases} -2x+3 & x < 0 \\ 3 & 0 \le x \le 3 \\ 2x-3 & x > 3 \end{cases}$$

We can sketch each part of this and then overlay the line for $f(x)$.