Rational functions and graphs

Algebra 5

INU0114/514 (Maths 1)

Dr Adrian Jannetta MIMA CMath FRAS

INTO Newcastle University

Objectives

In this lecture (and next seminar) we will do the following:

- Examine the behaviour of the reciprocal function $y = x^{-1}$.
- Understand how a function behaves close to an asymptote.
- Learn some notation associated with limits.
- Learn how to graph some rational functions.

Understanding the behaviour of $y = \dfrac{1}{x}$ is crucial to understanding how rational functions behave.

Use the interactive Geogebra app at

`http://www.geogebratube.org/student/m289915`

to help with this. The Desmos Graphing Calculator

`https://www.desmos.com/calculator`

is also a great way to graph rational functions.

Recap: degree of a polynomial

The *degree* of a polynomial is the highest power of that polynomial.

$$x^3 + 3x - 20 \qquad \text{has degree 3}$$
$$1 + 3x - 10x^3 - x^6 \qquad \text{has degree 6}$$

Fractions where the numerator degree is smaller than the denominator are *proper*.

If the numerator degree is greater than or equal to the degree of the *denominator* then the fraction is *improper*.

$$\frac{10}{x+3} \qquad \text{(proper)}$$
$$\frac{x^2+1}{x^2-9} \qquad \text{(improper)}$$

Improper fractions can be expressed using proper fractions by doing polynomial division.

Rational functions

A rational function is any function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

Where P and Q might both be functions and where $Q(x) \neq 0$.

- We are not interested in plotting tables of values for x and y; we might not choose points that show the important behaviour of the function.
- We will learn to sketch $f(x)$ so that behaviour over a suitable interval can be examined.

Rational functions show unusual behaviour close to values which make $Q(x) = 0$.

The "simplest" rational function to study is the *reciprocal function* $f(x) = \frac{1}{x}$.

The reciprocal function

Consider the reciprocal function

$$y = \frac{1}{x} \quad , \quad x \neq 0$$

Although y is not defined at $x = 0$ we can see what happens when x is close to 0.

What happens to y when x is positive, but approaching zero? We can calculate:

x	1	0.1	0.01	0.001	0.0001	...	0
$y = 1/x$	1	10	100	1000	10000	...	–

Mathematically, we would write this information as

$$x \to 0^+ \text{ then } y \to \infty$$

This is read as “x approaches 0 from above then y will tends towards infinity”.

The reciprocal function

$$y = \frac{1}{x} \quad , \quad x \neq 0$$

And what happens when x is negative, but approaching zero?

x	-1	-0.1	-0.01	-0.001	-0.0001	...	0
$y = 1/x$	-1	-10	-100	-1000	-10000	...	$-$

We could express this behaviour as

$$x \to 0^{-} \text{ then } y \to -\infty$$

This is read as “x approaches 0 from below then y tends towards negative infinity”.

The reciprocal function:

$$y = \frac{1}{x} \quad , \quad x \neq 0$$

Now think about what happens when x becomes a very large, positive value:

x	1	10	100	1000	10000	...
$y = 1/x$	1	0.1	0.01	0.001	0.0001	...

We can express this behaviour like this:

$$x \to \infty \text{ then } y \to 0^+$$

"As x tends to infinity then y approaches 0 from above".

The reciprocal function:

$$y = \frac{1}{x} \quad , \quad x \neq 0$$

We can think about x becoming a large, negative value:

x	-1	-10	-100	-1000	-10000	...
$y = 1/x$	-1	-0.1	-0.01	-0.001	-0.0001	...

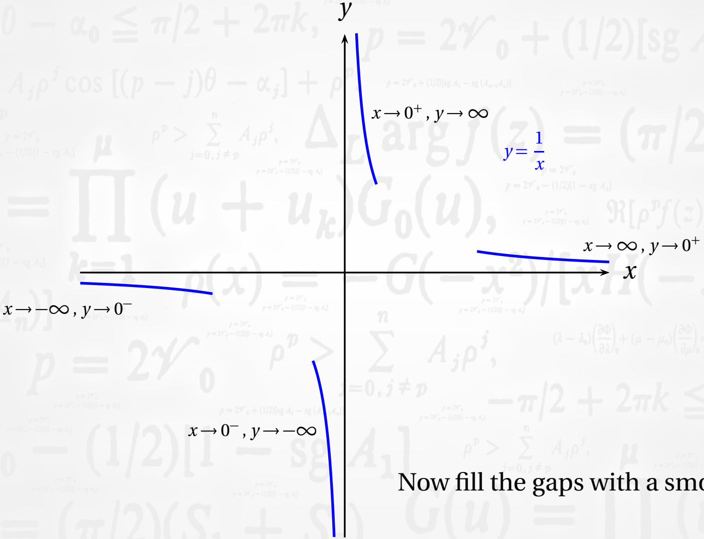
In mathematical notation:

$$x \to -\infty \text{ then } y \to 0^-$$

"As x tends to negative infinity then y approaches 0 from below".

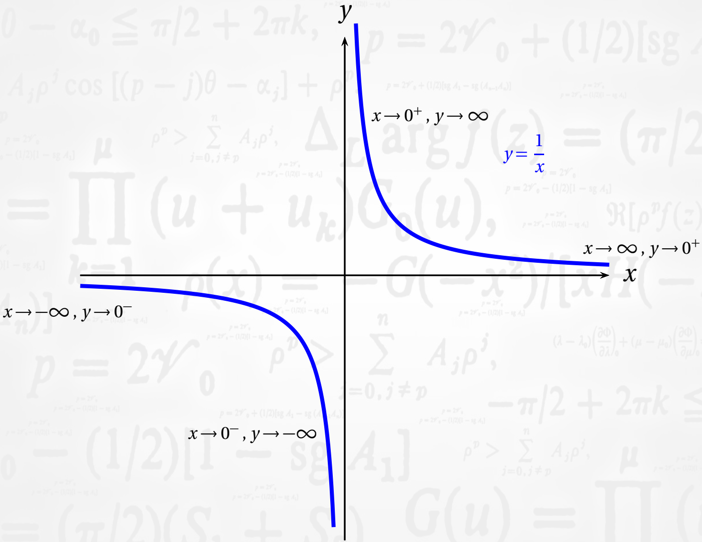
Now: assemble this information on a graph...

Sketching the graph of $f(x)=\dfrac{1}{x}$



Now fill the gaps with a smooth curve!

Sketching the graph of $f(x) = \frac{1}{x}$



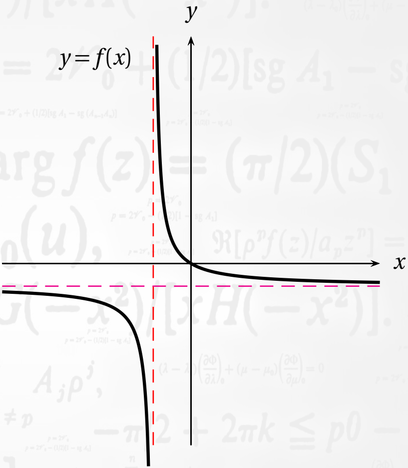
Asymptotes

Graphs of rational functions usually have asymptotes.

An *asymptote of a curve* is a line such that the distance between the curve and the line approaches zero as they tend to infinity.

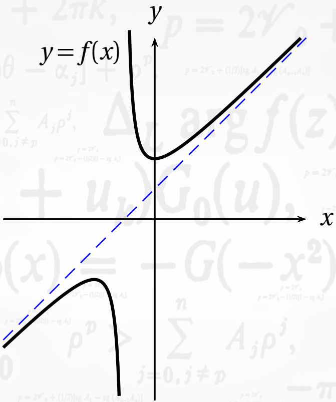
There are several types of asymptote to consider.

- Vertical asymptotes, $x = a$.
- Horizontal asymptotes, $y = a$
- Slant asymptotes...



Graph of a function showing two types of asymptote.

Some rational functions will show oblique (or slant) asymptotes.



Slant asymptotes have the general form $y = mx + c$.

Sketching rational functions

Some guidelines for sketching rational functions:

- Draw the axes of the graph.
- Solve $Q(x) = 0$ to find any vertical asymptotes.
- Find intersection points with the axes.

When numerator degree < denominator:

- Horizontal asymptote at $y = 0$

When numerator degree = denominator:

- Horizontal asymptote at $y = k$.

When numerator degree > denominator degree:

- Do polynomial division to find the oblique asymptote.

You may have noticed that vertical asymptotes correspond to domain restrictions and horizontal asymptotes correspond to range restrictions.

After that we'll investigate how the curve behaves as it approaches the asymptotes.

- Vertical asymptotes. What happens to the curve as it approaches from the left or right?
- Horizontal asymptotes. What happens as x becomes big (positive or negative)? Is the curve above or below the asymptote?

These guidelines don't have to be followed in order. Sometimes it may not be necessary (or possible) to use them all.

Also, differentiation can be used to look for turning points on the graph — more about this Maths 2 soon.

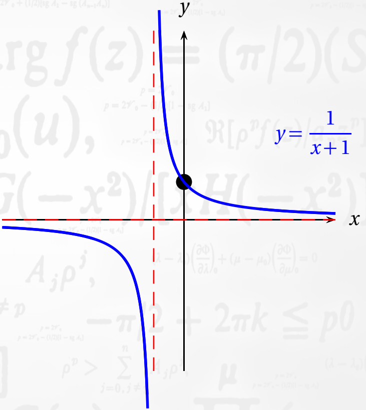
Rational function sketch

Sketch the graph of $y = \dfrac{1}{x+1}$.

- Draw the graph axes.
- Vertical asymptote at $x = -1$.
- What happens as x becomes big?
  Horizontal asymptote at $y = 0$.
- When $x = 0$, then $y = 1$.

Examine the asymptotes:

- As $x \to -1^+$ then $y \to -\infty$.
  As $x \to -1^-$ then $y \to \infty$.
- As $x \to \infty$, then $y \to 0^+$.
  As $x \to -\infty$, then $y \to 0^-$.
- Finish the curve!



Sketching a rational function

Sketch the graph of the function

$$f(x) = \frac{x}{x-1}$$

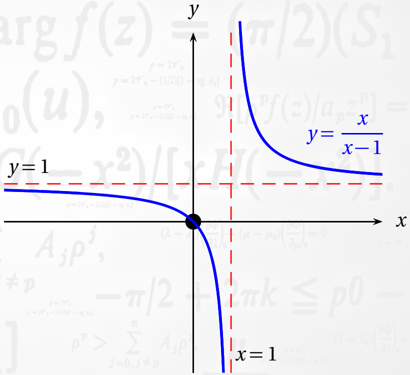
- Draw the axes.
- Vertical asymptote at $x = 1$.
- Horizontal asymptote: examine the highest powers on top and bottom.

  $$y \sim \frac{x}{x} = 1$$

  So, horizontal asympote at $y = 1$.
- Intersections: when $x = 0$ then $y = 0$ (i.e. the origin).

Examine the asymptotes:

- As $x \to 1^+$ then $y \to \infty$.
  As $x \to 1^-$ then $y \to -\infty$.
- What happens when x is large (positive and negative)?.
- Complete the curve.



Oblique (Slant) asymptotes

If the degree of the numerator is greater than the degree of the denominator then we'll get a slant asymptote.

Equations of slant asymptotes are found with polynomial division.

Sketching a rational function

Sketch the graph of

$$f(x) = \frac{x^2+1}{x-1}$$

There is a vertical asymptote at $x = 1$.

With long division we can write $f(x)$ as

$$f(x) = x + 1 + \frac{2}{x-1}$$

Now, as $x \to \infty$ then $f(x) \to x + 1$.

So, the slant asymtote is $y = x + 1$.

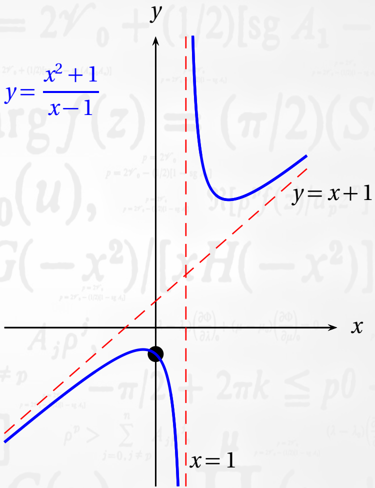
The rest of the analysis is the same as before.

For the function $f(x) = \dfrac{x^2+1}{x-1} = x+1+\dfrac{2}{x-1}$

- Draw the axes.
- Vertical asymptote at $x=1$.
- Slant asymptote: $y=x+1$.
- At $x=0$, we get $y=-1$.

Examine the asymptotes:

- As $x \to 1^+$ then $y \to \infty$.
  As $x \to 1^-$ then $y \to -\infty$.
- When x is large, $y \to x+1$ (it approaches the slant asymptote).
- Complete the curve.



The curve never crosses an asymptote.
That fact helps us to get the curve position and direction correct.

Sketching a rational function

Sketch the graph of the function

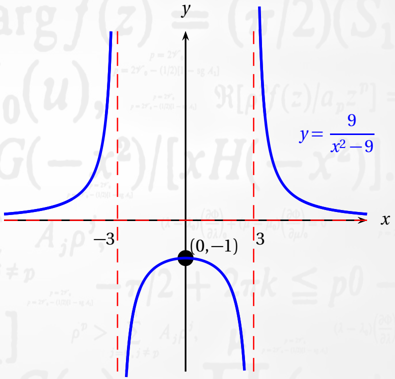
$$f(x) = \frac{9}{x^2 - 9} = \frac{9}{(x+3)(x-3)}$$

Sketch the axes...

- Vertical asymptotes at $x = \pm 3$
- Horizontal asymptote at $y = 0$.
- When $x = 0$, then $y = -1$

Check the curve near the asymptotes...

- As $x \to 3^+$ then $y \to \infty$.
  As $x \to 3^-$ then $y \to -\infty$.
- As $x \to -3^+$ then $y \to -\infty$.
  As $x \to -3^-$ then $y \to \infty$.
- As $x \to \infty$ then $y \to 0^+$.
  As $x \to -\infty$ then $y \to 0^+$.
- Connect the pieces - through the point - and without crossing asymptotes.



Hole in a line

Sketch the graph of the function

$$f(x) = \frac{x^2 - x - 2}{x - 2}$$

The domain of this function is not defined for $x = 2$.

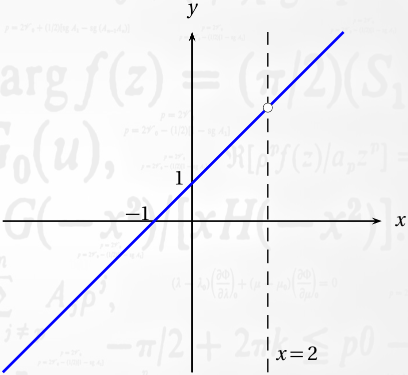
Also, notice that the function can be simplified:

$$f(x) = \frac{(x+1)(x-2)}{x-2} = x + 1$$

This is a straight line. But we must still keep the domain restriction.

The function graph is a straight line with an infinitely small hole!

$$f(x) = x + 1\,,\; x \neq 2$$



Also, because of the domain restriction, the function has a restricted range: $f(x) \neq 3$.