INVERSE FUNCTIONS

ALGEBRA 4

INU0114/514 (MATHS 1)

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Objectives

In this session we will do the following:

- Definition of an inverse function.
- Find the inverse function of a given function.
- Sketching an inverse function.
- Finding an inverse function by restricting the domain of a function.
- Inverse trig functions.
- The exponential and logarithmic functions.

Composite functions: recap

Special case of a composite function

Given the functions

$$f(x) = 4x + 5 \quad \text{and } g(x) = \tfrac{1}{4}(x - 5)$$

Find the composite functions $fg(x)$ and $gf(x)$.

The first composite function is

$$\begin{aligned} fg(x) &= 4\left(\tfrac{1}{4}(x-5)\right)+5 \\ &= x-5+5 \\ fg(x) &= x \end{aligned}$$

And the other is

$$\begin{aligned} gf(x) &= \tfrac{1}{4}((4x+5)-5) \\ &= \tfrac{1}{4}(4x) \\ gf(x) &= x \end{aligned}$$

In each case x is mapped to the domain of a second function, and the second function returns it to x again. Pairs of functions which do this are called *inverse functions*.

Inverse functions

Suppose we have a function f which takes an input value x and maps it to a value y

$$f(x) = y \tag{1}$$

An inverse function, written f^{-1}, is one that will accept y as input and map it back to x

$$f^{-1}(y) = x \tag{2}$$

The inverse function *must* generate values in the same domain as the original function. Otherwise it is not an inverse function.

An inverse function 'undoes' the result of the original function.

Also, we've shown that

$$ff^{-1}(x) = f^{-1}f(x) = x$$

Finding the inverse function

Inverse by rearranging

Find the inverse of of $f(x) = 2x + 4$.

Let $y = 2x + 4$.

Rearrange to make x the subject:

$$\begin{aligned} y - 4 &= 2x \\ \frac{y-4}{2} &= x \end{aligned}$$

Switch the labels of x and y around:

$$\frac{x-4}{2} = y$$

The inverse function of $f(x)$ is given by

$$f^{-1}(x) = \frac{x-4}{2}$$

Inverse function

Find the inverse of $f(x) = \dfrac{3x-7}{2}$.

First, write it as $y = \dfrac{3x-7}{2}$.

Rearrange this to make x the subject:

$$\begin{aligned} 2y &= 3x-7 \\ 2y+7 &= 3x \\ \therefore \frac{2y+7}{3} &= x \end{aligned}$$

Finally swap the positions of x and y on each side

$$\frac{2x+7}{3} = y$$

The inverse function is

$$f^{-1}(x) = \frac{2x+7}{3}$$

The reason why we are allowed to swap x and y to get the inverse function will be explained later.

Finding an inverse function

Find the inverse function to

$$f(x) = \frac{x}{x-1} \quad , x \neq 1$$

First, write the function as $y = \dfrac{x}{x-1}$.

Rearrange to make x the subject:

$$\begin{aligned} y(x-1) &= x \\ xy - y &= x \\ xy - x &= y \\ x(y-1) &= y \\ x &= \frac{y}{y-1} \end{aligned}$$

Therefore the inverse function is

$$f^{-1}(x) = \frac{x}{x-1} \quad , x \neq 1$$

In this case, the function and its inverse are the same.

Domain and range of inverse functions

Domain and range for inverse functions

Given the function and its inverse:

$$f(x) = \frac{1}{x+1} \quad \text{and } f^{-1}(x) = \frac{1}{x} - 1$$

Write down the domain and range of each function.

Assembling these results into a table we can see some connections:

	$f(x)$	$f^{-1}(x)$
Domain	$x \neq -1$	$x \neq 0$
Range	$f(x) \neq 0$	$f^{-1}(x) \neq -1$

This illustrates some general features of inverse functions.

- The domain of the $f(x)$ is the same as the range of the inverse function $f^{-1}(x)$.
- The range of $f(x)$ is the same as the domain of the inverse function $f^{-1}(x)$.
- The point (x, y) in $f(x)$ is mapped to the point (y, x) of $f^{-1}(x)$.

(The last point is the reason why we can swap x and y in the rearrangement method).

Domain and range for inverse functions

Given the function:

$$f(x) = \frac{3}{2x-5}$$

Find $f^{-1}(x)$ and write down its domain and range.

The inverse function is

$$f^{-1}(x) = \frac{3}{2x} + \frac{5}{2}$$

The domain of $f(x)$ is $x \neq \frac{5}{2}$. Therefore the range of the inverse function is

$$f^{-1}(x) \neq \frac{5}{2}$$

The range of $f(x)$ is $f(x) \neq 0$. The domain of the inverse function must be

$$x \neq 0$$

To summarise:

$$f^{-1}(x) = \frac{3}{2x} + \frac{5}{2} \quad , x \neq 0,\ f^{-1}(x) \neq \frac{5}{2}$$

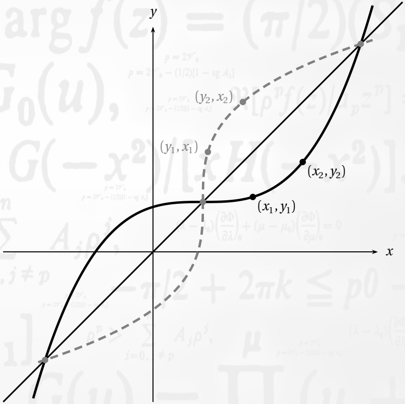
Graph of an inverse function

The graph of an inverse function is a reflection in the line $y = x$.

A point on the function curve has coordinates (x, y). The equivalent point on the inverse function curve has coordinates (y, x).

Here's a rough guide to sketching an inverse function:

1. Sketch the graph of $y = f(x)$.
2. Draw the line $y = x$.
3. If points (x_1, y_1) and (x_2, y_2) are on the curve $f(x)$ then (y_1, x_1) and (y_2, x_2) will be on the curve $f^{-1}(x)$.
4. Look for points where $f(x)$ intersects $y = x$; the inverse function will also intersect at the same place.
5. Draw the curve through the points.



Graph of a simple inverse function

Given that $f(x) = 2x + 1$ sketch the graph of $f^{-1}(x)$.

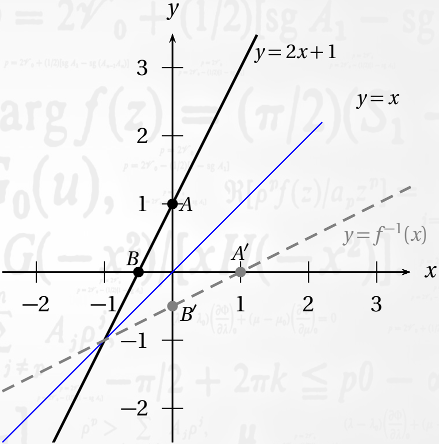
Sketch the function $y = 2x + 1$ on a graph then add the line $y = x$. Try to do this as accurately as possible.

Use the intersection points to help with the sketch.

The point $A(0,1)$ will be mapped to $A'(1,0)$.

The point $B(-\frac{1}{2}, 0)$ will be mapped to $B'(0, -\frac{1}{2})$.

Draw a straight line through the points.



It's easy to check that the inverse function is $f^{-1}(x) = \frac{1}{2}x - \frac{1}{2}$.

Sketch an inverse function

Given the function $f(x) = x^2$, $x \geq 0$, sketch the inverse function.

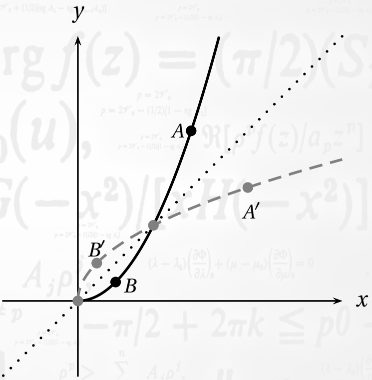
Make a sketch of $y = x^2$ for positive (and zero) values of x.

Add the line $y = x$ to the graph.

Look for places where $f(x)$ crosses the straight line; these are places where $f^{-1}(x)$ will also cross the straight line.

On the graph, the points A' and B' are the reflections of the points A and B respectively.

By plotting enough of these points you can build up a picture of the inverse function.

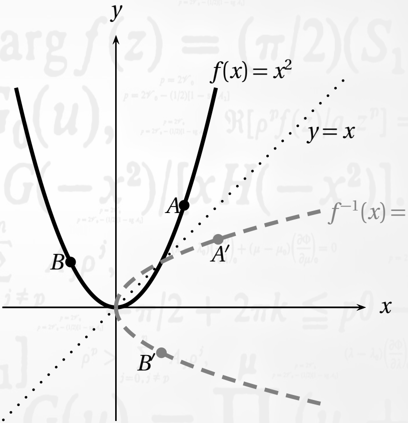


In the previous example we graphed the inverse function of

$$f(x) = x^2 \, , \, x \geq 0$$

Why did we restrict the domain?

Without the restriction our graph would look like this one.



Use the method given previously to construct the inverse.

But the 'inverse' is not actually a function — it fails the vertical line test.

It is a one-to-many relation. Each x value doesn't give a unique y value — unless we apply the domain restriction.

Some functions do not have inverse functions unless the domain is restricted.

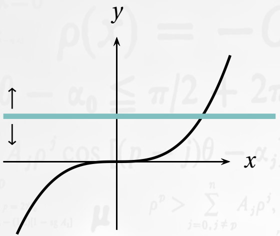
Horizontal line test

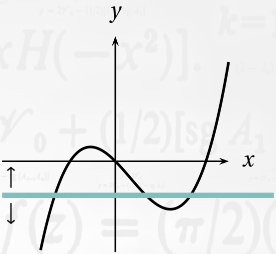
The horizontal line test is a simple graphical test that will determine whether a given function has an inverse function.

Make a horizontal line (e.g. with a ruler) on the graph of $y = f(x)$.

An inverse function $f^{-1}(x)$ exists only if the line touches one point on the curve when the line is moved vertically (up and down).

If it touches more than one point then the inverse can be found if domain restrictions are used to define the function so that only one point is touched.





In this example, a horizontal line would only ever touch the curve at one point. An inverse function $f^{-1}(x)$ could be found by rearrangement of the function or by reflection on a sketch.

The inverse function for this curve does not exist: a horizontal line would sometimes touch three places on the curve.
Domain restrictions would be necessary to define an inverse function.

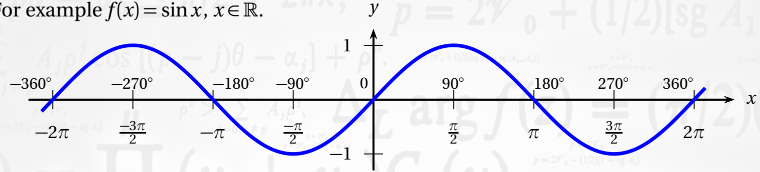
Functions which are one-to-one maps don't need further domain restrictions before an inverse function can be defined.

Functions which are many-to-one maps do need further domain restrictions before an inverse function can be defined.

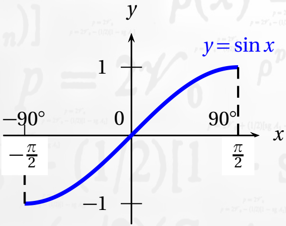
Inverse trigonometric functions

The graphs of sine, cosine and tangent are *many-to-one* relations.

For example $f(x) = \sin x,\ x \in \mathbb{R}$.



Applying the *horizontal line test* to the graphs of those functions shows that the inverse cannot exist unless domain restrictions are given.



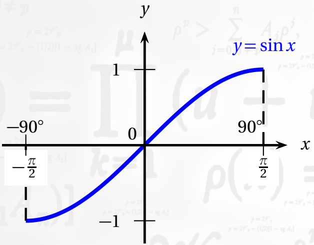
The domain restriction for sine is $-90° \leq x \leq 90°$.

Or in radians: $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

The sine graph over this restricted domain *does* have an inverse.

Inverse sine function

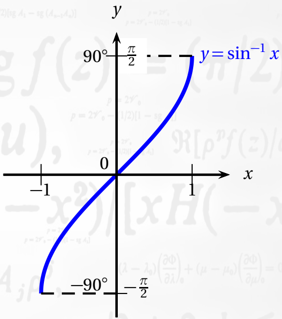
$f(x) = \sin x$



Domain: $-90° \leq x \leq 90°$

or $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

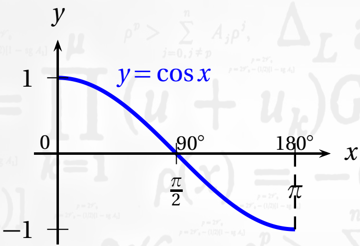
$f^{-1}(x) = \sin^{-1} x.$



Domain: $-1 \leq x \leq 1$

Inverse cosine function

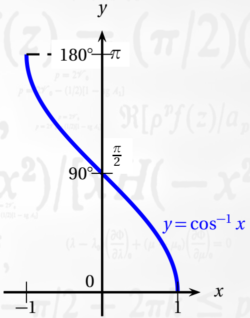
$f(x) = \cos x$



Domain: $0° \leq x \leq 180°$

or $0 \leq x \leq \pi$

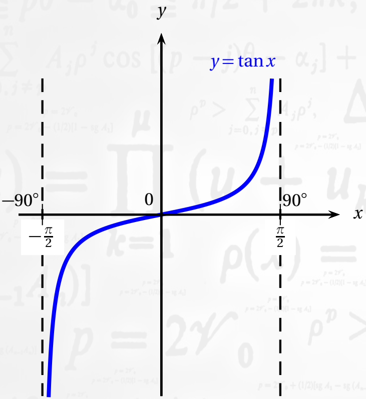
$f^{-1}(x) = \cos^{-1} x.$



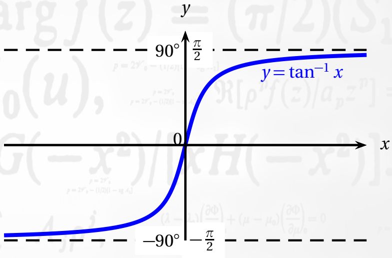
Domain: $-1 \leq x \leq 1$

Inverse tangent function

$$f(x) = \tan x$$



$$f^{-1}(x) = \tan^{-1} x$$



Domain: $-90° \leq x \leq 90°$

or $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

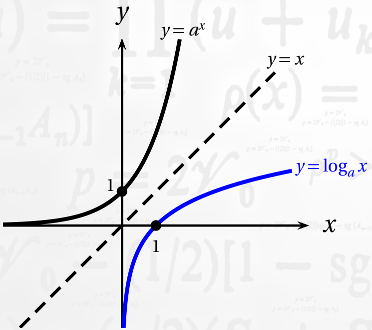
Domain $x \in \mathbb{R}$

Exponential and logarithmic functions

The exponential and logarithm functions are related through

$$a^x = y \text{ and } \log_a y = x$$

This definition is the same as equations (1) and (2) from earlier. It means that the logarithm function is the inverse of the exponential function. And vice versa.



The graph of each function is a reflection of the other in the line $y = x$.

The exponential function a^x has domain $x \in \mathbb{R}$. The range is $f(x) > 0$.

The domain of $\log_a x$ is $x > 0$. The range is $f(x) \in \mathbb{R}$.

Test yourself...

You should be able to solve the following problems if you have understood everything in these notes.

1. Given $f(x) = 1 - 3x$ write down $f^{-1}(x)$.
2. Given $g(x) = 2x$, $-4 < x < 3$, write down $f^{-1}(x)$ and state its domain.
3. Does the function $f(x) = x^2 + x + 1$, $x \in \mathbb{R}$ have an inverse?
4. Does the function $f(x) = x^2 - 2x + 1$, $x \in \mathbb{R}$, $x > 1$ have an inverse?

Answers:

1. $f^{-1}(x) = \frac{1}{3}(1 - x)$
2. $f^{-1}(x) = \frac{1}{2}x$, $-8 < x < 6$.
3. No! $f(x)$ fails the horizontal line test.
4. Yes! $f(x)$ passes the horizontal line test.