Composite Functions

Algebra 4

INU0114/514 (Maths 1)

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Objectives

In this session we will do the following:

- Construct composite functions from two simple functions
- Evaluate a composite functions and solve equations with them.
- Determine the domain and range of a composite function
- Decompose a function into two simpler functions

Composite functions

Consider two functions

$$f(x) = x^2 \quad \text{and } g(x) = x + 3$$

A new function fg can be constructed by substituting $g(x)$ into $f(x)$.

$$fg(x) = f(g(x)) = (x+3)^2$$

Similarly, we can compose the function $gf(x)$ by substituting $f(x)$ into $g(x)$:

$$gf(x) = g(f(x)) = x^2 + 3$$

The functions $fg(x)$ and $gf(x)$ are called *composite functions* (or *function of a function*).

We might also write a composite function as $f \circ g$ or fg.

The order of composition matters. It is generally the case that $fg(x) \neq gf(x)$.

Composite function

Given that $f(x) = 2x - 1$ and $g(x) = x^2 + x + 2$ find $fg(x)$ and $gf(x)$.
Also, evaluate $gf(2)$.

In this case we find

$$\begin{aligned} fg(x) &= 2(x^2 + x + 2) - 1 \\ &= 2x^2 + 2x + 4 - 1 \\ \therefore fg(x) &= 2x^2 + 2x + 3 \end{aligned}$$

And

$$\begin{aligned} gf(x) &= (2x-1)^2 + (2x-1) + 2 \\ &= 4x^2 - 4x + 1 + 2x - 1 + 2 \\ \therefore gf(x) &= 4x^2 - 2x + 2 \end{aligned}$$

Therefore $gf(2) = 4(2^2) - 2(2) + 2 = 14$.

Equation with a composite function

Given the functions

$$f(x) = 4x + 1\,,\ x \in \mathbb{R} \quad \text{and } g(x) = x^3\,,\ x \in \mathbb{R}$$

Solve $gf(x) = 125$.

The composite function is

$$gf(x) = (4x + 1)^3$$

Therefore we solve:

$$(4x + 1)^3 = 125$$

Take cube-roots:

$$4x + 1 = 5$$

Solve to get $x = 1$.

Equation with a composite function

Given the functions

$$f(x) = \frac{4x+2}{5}, x \in \mathbb{R} \quad \text{and } g(x) = 2^x, x \in \mathbb{R}$$

Solve $fg(x) = 26$.

The composite function is

$$fg(x) = \frac{4(2^x)+2}{5}$$

Therefore we solve:

$$\frac{4(2^x)+2}{5} = 26$$

Rearrange to get

$$2^x = 32$$

Solve to get $x = 5$.

Domains and range of a composite functions

Consider the function $gf(x)$ formed by combining $f(x)$ and $g(x)$.

In picture form we can represent the composite as:

$$\underset{\text{domain of } f}{x} \longmapsto \boxed{\text{function } f} \longmapsto \overset{\text{range of } f}{\underset{\text{domain of } g}{f(x)}} \longmapsto \boxed{\text{function } g} \longmapsto \underset{\text{range of } g}{g(x)}$$

Any restrictions on the domain of f will have consequences for the numbers which become the domain of g, and for the numbers which emerge in the range of g.

The domain of a composite function must be considered carefully.

Consider the functions $f(x) = -x^2$ and $g(x) = \ln x$.

What is the composite function $gf(x)$?

In this case we could simply write:

$$gf(x) = \ln(-x^2)$$

Since the range of $f(x) < 0$ then $g(x)$ is not defined for any of those values of (its domain is $x > 0$) then the function $gf(x)$ does not exist.

Some pairs of functions cannot be composed into a new function.

Other pairs of functions might require us to restrict the domain of the first function in order to make a composite function.

Composite function: domain and range

Given the functions

$$f(x) = 2x - 3 \quad \text{and } g(x) = \sqrt{x}$$

Construct the composite function $gf(x)$.

The input function $2x-3$ accepts all real numbers as input. The domain is $x \in \mathbb{R}$.

The composite function is

$$gf(x) = \sqrt{2x-3}$$

Clearly this function has a restricted domain for which $2x-3 \geq 0$. We solve this to get $x \geq \frac{3}{2}$.

The range of the function contains only nonnegative numbers. Therefore, the completely specified function, using the more restricted domain, is:

$$gf(x) = \sqrt{2x-3}\ ,\ x \geq \tfrac{3}{2}\ ,\ gf(x) \geq 0$$

The domain of a composed function is either the same as the domain of the first function, or else lies inside it.

The range of a composed function is either the same as the range of the second function, or else lies inside it

In a composite function, the output (range) of one function will become the input (domain) of another.

If a function $f(x)$ is defined with restrictions on its domain, then those restrictions may influence the domain of the other function $g(x)$ and also its range.

Composite function: domain and range

Given the functions

$$f(x) = \frac{1}{x+2} \quad \text{and } g(x) = \frac{x}{x-3}$$

What is the domain of $fg(x)$?

The domain of the input $g(x)$ accepts all real numbers except $x = 3$. That means the composite fg will not accept $x = 3$.

The domain of $f(x)$ has the restriction $x \neq -2$. That means any values from g which make an output of -2 must be rejected. To find the values, solve:

$$\frac{x}{x-3} = -2 \qquad \therefore x = 2$$

The composite function has domain $x \in \mathbb{R}$, $x \neq 2$, $x \neq 3$.

Procedure for finding the domain

Notice that the composite function in the previous example was

$$fg(x) = \frac{x-3}{3(x-2)}$$

One of the domain restrictions came from $g(x)$ (the input) and the other is seen by examining $fg(x)$ (the composite).

In general a safe procedure for finding the domain of a composite function consists of two steps:

1. Find the domain of the input function. If there are any restrictions — keep them!
2. Construct the composite function and find its domain. Note any restrictions on this domain and add them to those found in step (1). If there is overlap (intersection) then use the most restricted domain.

It's possible that the domain of the composite function will bear little resemblence to the domains of the original functions.

Domain and range of a composite function

Given the functions

$$f(x) = -x \text{ and } g(x) = \ln x$$

Write down the domain and range of $gf(x)$.

The domain of the input function $-x$ is all the real numbers ($x \in \mathbb{R}$).

The composite function is

$$gf(x) = \ln(-x)$$

This function will only exist when $(-x)$ is positive. In other words, if the domain is $x < 0$.

In that case the range of the logarithm function is all real numbers.

Therefore, using the restricted domain, we have

$$gf(x) = \ln(-x) \quad , x < 0 \, , gf(x) \in \mathbb{R}$$

Domain and range of a composite function

Given the functions

$$f(x) = x^2 + 2 \qquad \text{and} \qquad g(x) = \sqrt{3 - x}$$

Find the domain and range of the composite function $fg(x)$.

The input function $g(x)$ has a restricted domain; it is restricted to $x \leq 3$.

The composite function is:

$$\begin{aligned} fg(x) &= (\sqrt{3-x})^2 + 2 \\ &= 3 - x + 2 \\ \therefore fg(x) &= 5 - x \end{aligned}$$

This composite function has no further restrictions - it can accept any real number as an input.

However, we keep the restriction from the input function so that $fg(x) = 5 - x\,,\ x \in \mathbb{R},\ x \leq 3$.

Given that domain restriction, then the range will also be restricted: only values greater than or equal to 2 will be generated.

$$\therefore fg(x) = 5 - x\,,\ x \in \mathbb{R},\ x \leq 3\,,\ fg(x) \geq 2$$

Test yourself...

You should be able to solve the following problems if you have understood everything in these notes.

1. Given $f(x) = 1 - x^2$ and $g(x) = x^3$ write down $fg(x)$ and $gf(x)$.
2. Evaluate $fg(2)$ and $gf(0)$ for the functions in part (1).
3. Given $f(x) = x^2$, $0 < x < 8$ and $g(x) = 4x$, $x \in \mathbb{R}$
   Write down $gf(x)$ with domain and range.
4. Given that $f(x) = 2x$, $x \neq 3$ and $g(x) = x + 5$, what is the domain and range of $gf(x)$?

Answers:

1. $fg(x) = 1 - x^6$ and $gf(x) = (1 - x^2)^3$
2. $fg(2) = -63$ and $gf(0) = 1$.
3. $gf(x) = 4x^2$, $0 < x < 8$, $0 < gf < 256$.
4. $gf(x) = 2x + 5$, $x \neq 3$, $f(x) \neq 11$.

Decomposing functions

In some calculus applications (e.g. the chain rule for differentiation) it is useful to reverse the process studied here. We must *decompose* a function into two simpler functions.

For example, given the function $h(x) = (4x-3)^3$ find two simpler basis functions $f(x)$ and $g(x)$.

Given that $h(x) = gf(x)$ then it must be the case that

$$f(x) = 4x - 3 \text{ and } g(x) = x^3$$

Decomposing a function

Decompose the function $h(x) = \sin^3 x$ into two simpler functions.

In this case the function is $h(x) = (\sin x)^3$ so that $f(x) = \sin x$ and $g(x) = x^3$.

In which case $h(x) = gf(x)$.