FUNCTIONS

ALGEBRA 4

INU0114/514 (MATHS 1)

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Overview

On the course we have used the word “function” when dealing with algebra and with graphs. But mathematicians have a very formal (strict!) definition for the word function and it’s now time we got used to it.

In this session we will do the following:

- Understand one-to-one, one-to-many and many-to-one mappings.
- Understand the formal definition of a function.
- Understand the definitions of domain and range.
- Know how to apply the vertical line test.
- Be able to determine the domain and range of given functions

Background

Nikolai Lobachevsky (1792 — 1856)

The definition of a function has changed over time. Mathematicians didn't formalise a definition for a long time.

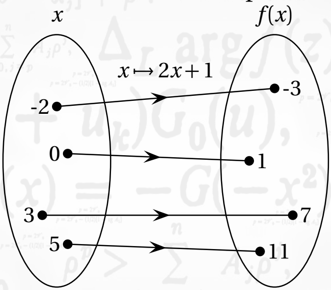
Johan Dirichlet (1805 — 1859) of Germany and Nikolai Lobachevsky (1792 — 1856) of Russia gave us our modern definition of function. Every member of a set is related to just one corresponding member in another.

We will study this precise definition in detail next.

Definition of a function

A function is a rule which links an input number to an output number.

The mathematical definition of a function is very strict; there can only be one output number associated with each input number.



To define a function we need a rule and a set of values.

Functions are sometimes called maps, mappings, transformations or operators.

Function notation

Consider a function rule which takes a number, multiplies it by 3 and adds 5 to the result. This can be represented using the notation:

$$f : x \mapsto 3x + 5 \tag{1}$$

A shorter notation (which you are probably familiar with) is:

$$f(x) = 3x + 5 \tag{2}$$

We can evaluate the function for a given input value:

$$f(3) = 14$$

When the value 3 is used as input, then the value 14 is output.

We will use this notation often but you should be aware that functions can be written in form given in (1). Another way to give the function:

$$y = 3x + 5$$

In a function definition like this then x it is called the *independent variable* and $f(x)$ or y is the *dependent variable*.

Domain of a function

The domain of a function is the list of all possible values that the independent variable (input values) can take.

Consider $f(x) = 3x + 5$

The domain consists of the values of x that we can put into the function. In this case, we can use any value we want. There are no restrictions — x can be any real number.

The domain, when it is given, is usually presented alongside the function like this:

$$f(x) = 3x + 5, \qquad x \in \mathbb{R}$$

The expression $x \in \mathbb{R}$ is read as "x is a member of the set of real numbers".

Other functions may have restrictions on the domain.

For example, the logarithm function can only accept positive values as input:

$$f(x) = \ln x, \qquad x > 0$$

Or the reciprocal function, which cannot accept the value of zero:

$$f(x) = \frac{1}{x}, \quad x \neq 0$$

When there are no restrictions on the domain (or when the restrictions are obvious!) then the domain may be ommitted.

The USS Yorktown was a warship in the United States Navy and used to test new technology in the 1980s. It utilised 27 Pentium Pro chips and Windows NT software to control many onboard systems.

In 1987 the computers controlling the propulsion system of the ship crashed when someone put a zero into a database.

The programmers of the software had not taken care in specifying a function domain and a division by zero occurred.

The resulted in a very costly and time consuming mistake: the ship in harbour was stranded for hours!

Range of a function

The range of a function is the list of all possible values that the dependent variable (output values) can take.

Consider the function

$$f(x) = 3x + 5, \qquad x \in \mathbb{R}$$

This function accepts any real number x as input. The output, $f(x)$, will also be another real number.

The complete function definition with domain and range:

$$f(x) = 3x + 5, \quad x \in \mathbb{R}, f(x) \in \mathbb{R}$$

The function $f(x) = x^2$ accepts any real number.

But the output value can only be positive or negative. There is a restriction on the range:

$$f(x) = x^2, \quad x \in \mathbb{R}, f(x) \geq 0$$

Domain and range

Consider the function defined by $f(x) = x^2 - 9$. What are the domain and range of the function?

Consider the *domain*. This is the values that x can take. Here x can take any real value so the domain must be the set of all real numbers,

$$\text{Domain:} \;\; x \in \mathbb{R}$$

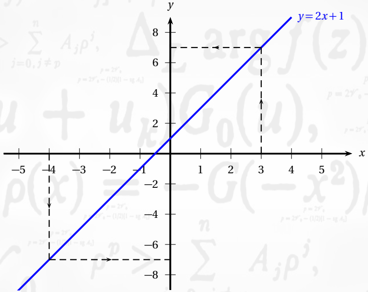
For the *range*, consider the expression $x^2 - 9$. We can see that $x^2 \geq 0$. So $x^2 - 9 \geq 0 - 9$, therefore $x^2 - 9 \geq -9$. However, $f(x) = x^2 - 9$ so:

$$\text{Range:} \;\; f(x) \in \mathbb{R}, \;\; f(x) \geq -9.$$

Graph of a function

The graph of a function can be useful for visualising the domain and range.

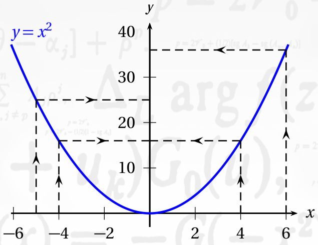
Plotted on a graph $y = 2x + 1$ looks like:



- Each value in the domain (x values) is mapped onto one value in the range (y values).
- Each value in the range is mapped to one value in the domain.
- This relationship is described as a *one-to-one* mapping.

Now consider the function defined by $y = x^2$.

The graph of this function looks like:



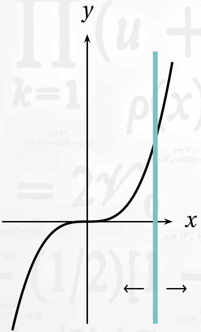
- Each value in the domain (x values) is mapped onto one value in the range (y values).
- Many values in the domain can be mapped to the same value in the range.
- This relationship is a *many-to-one* mapping.

Vertical line test

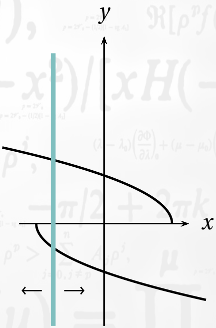
The vertical line test is a quick way to check of whether a particular graph represents a function or not.

If the graph represents a function then the vertical line will only ever touch the curve at one point.

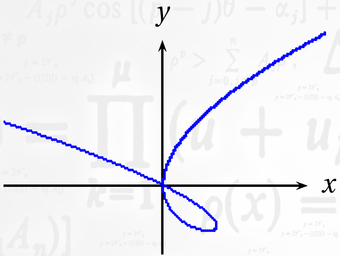
A function



Not a function

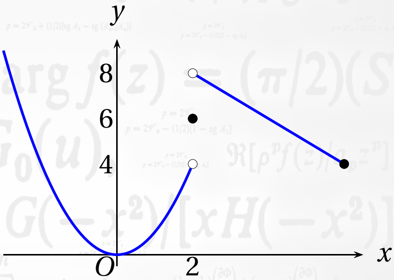


Is this a function?



No!

Is this a function?



Yes!

Function range

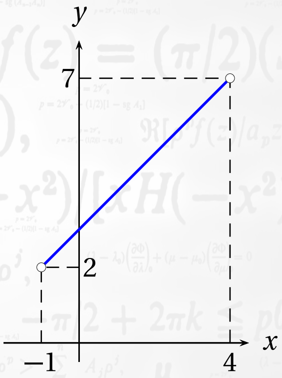
Find the range of $f(x) = 3 + x, \quad -1 < x < 4.$

When $x = -1$ then $f(-1) = 2$.

When $x = 4$ then $f(4) = 7$.

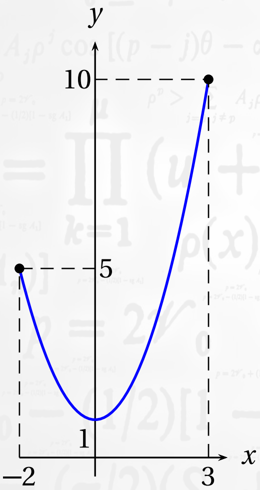
The range is $2 < f(x) < 7$.

For 1:1 functions we can use the end points of the domain to find the range.



Function range

Find the range of $f(x) = x^2 + 1, \quad -2 \leq x \leq 3$.



Refer to the graph opposite: we can't just use the end points of the domain.

When $x = -2$ then $f(-2) = 5$.

When $x = 3$ then $f(3) = 10$.

But: the minimum point is at $y = 1$.

The range is $1 \leq f(x) \leq 10$.

A sketch might be useful for finding the range!

Restrictions on the domain and range

Consider the relationship defined by

$$y^2 = x$$

Is this a function? We can see more clearly if we write it as

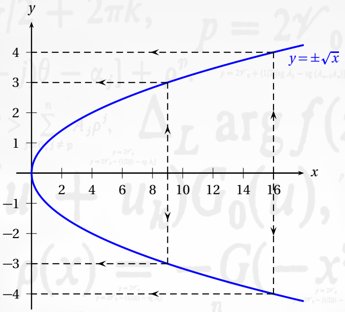
$$y = \pm\sqrt{x}$$

- If we assume $x \in \mathbb{R}$ then this is NOT a function. Some domain values $(x < 0)$ will not give a real value in the range.
- To make a function the *domain* must be restricted to the values $x \geq 0$.
- The equation becomes

$$y = \pm\sqrt{x}, \quad , x \geq 0$$

This is better but still NOT a function...

The graph below shows why $y = \pm\sqrt{x}$ is not a function:

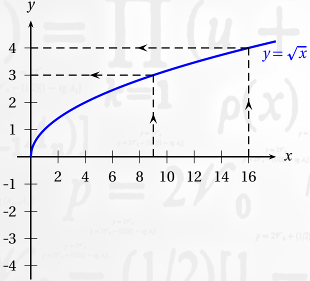


- Each domain value (x-axis) is mapped to more than one value in the range (y values).
- It is a *one-to-many* map.
- We need another restriction...

To convert the one-to-many map to a function we introduce a range restriction.

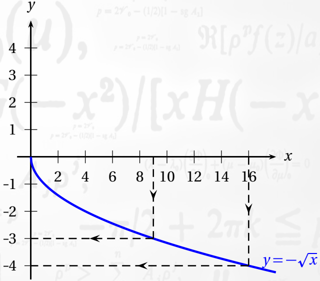
Restricting the *range* to exclude negative y-values:

$$y = +\sqrt{x}, \quad , x \geq 0, f(x) \geq 0$$



Restricting the *range* to exclude positive y-values:

$$y = -\sqrt{x}, \quad x \geq 0, f(x) \leq 0$$



Both relations satisfy the one input, one output definition of a function.

Domain and range restrictions

Consider the function defined by $f(x) = 3 + \frac{1}{x-2}$. What are the domain and range of this function?

Domain of the function. Consider $\frac{1}{x-2}$: this is valid except when $x - 2 = 0$, so $\frac{1}{x-2}$ is not defined when $x = 2$. The *domain* of the function $f(x)$ is:

$$x \in \mathbb{R}, \;\; x \neq 2$$

Range of the function. Consider $\frac{1}{x-2}$: the numerator is never equal 0, so $\frac{1}{x-2} \neq 0$, which means that $3 + \frac{1}{x-2} \neq 3$. So $f(x) \neq 3$. Therefore the *range* of the function is:

$$f(x) \in \mathbb{R}, \;\; f(x) \neq 3$$

Test yourself...

You should be able to solve the following problems if you have understood everything in these notes.

1. Write down the domain and range of $f(x) = \cos x$.
2. Write down the domain and range of $f(x) = 3^x$.
3. Given $f : x \mapsto x^2 + 1, -1 \leq x \leq 3$, what is the range of f ?
4. Given $f : x \mapsto 1 - 2x, -3 \leq f \leq 11$, what is the domain of f ?
5. Is it correct to call $f(x)$ a function, where $f(x) = \sqrt[3]{x}$, $x \in \mathbb{R}$?

Answers:

1. Domain: $x \in \mathbb{R}$, Range: $-1 \leq f(x) \leq 1$.
2. Domain: $x \in \mathbb{R}$, Range: $f(x) > 0$.
3. Range: $1 \leq f \leq 10$.
4. Domain: $-5 \leq x \leq 2$
5. Yes! Each x input will only produce one output.