Factor & Remainder Theorems

Algebra 3

INU0114/514 (Maths 1)

Dr Adrian Jannetta MIMA CMath FRAS

Overview

- Understand the factor theorem.
- Understand remainder theorem.
- Know how to use factor theorem to factorise simple cubic equations.

Motivation: roots of a polynomial

Consider the polynomial

$$f(x) = (x-1)(x-2)(x+3)$$

The *roots* of $f(x)$ are $x = 1$, $x = 2$ and $x = -3$.

Another way of thinking about this is that these values are the *solutions* to the equation $f(x) = 0$.

If we substitute any of the roots into $f(x)$ then the result is zero:

$$f(1) = f(2) = f(-3) = 0$$

In this example it was easy to see the roots!

Typically, we have an $f(x)$ that isn't factorised.

However, if we can find values $x = a$ for which $f(a)$ is zero then we'll know we've found a solution.

Factor theorem

Factor theorem

The *factor theorem* is useful for finding the factors of a polynomial. It states that the polynomial $f(x)$ has a factor $x-a$ only if $f(a)=0$.

The proof is simple. When polynomial division is carried with out with a linear divisor then

$$\frac{f(x)}{x-a} \equiv q(x) + \frac{r}{x-a}$$

where $q(x)$ is the quotient and r is the remainder.

If $x-a$ is a factor of $f(x)$ then the remainder will be zero (i.e. $r=0$) so that

$$\frac{f(x)}{x-a} = q(x)$$

Multiply both sides by $x-a$ to obtain:

$$f(x) = (x-a)q(x)$$

So when $x=a$ we have the obvious result:

$$f(a) = (a-a)q(x) = 0$$

And so $f(x)$ has a factor $x-a$ only if $f(a)=0$

Applying the factor theorem

Use the factor theorem to factorise

$$f(x) = x^3 - x^2 - 14x + 24$$

Let's try find simple linear factors.

Evaluate: $f(1) = 10$. Therefore $x - 1$ is not a factor of $f(x)$.

Evaluate: $f(2) = 0$. Therefore $x - 2$ is a factor of $f(x)$.

We can use long division or comparing coefficients to factorise further. For example, by comparing coefficients:

$$\begin{aligned} x^3 - x^2 - 14x + 24 &\equiv (x-2)(ax^2 + bx + c) \\ &\equiv ax^3 - 2ax^2 + bx^2 - 2bx + cx - 2c \\ x^3 - x^2 - 14x + 24 &\equiv ax^3 + (b-2a)x^2 + (c-2b)x - 2c \end{aligned}$$

Compare $[x^3]$: $a = 1$.

Compare $[x^2]$: $-1 = b - 2a$. Therefore $b = 1$

Compare $[x]$: $-14 = c - 2b$. Therefore $c = -12$.

$\therefore\ x^3 - x^2 - 14x + 24 \equiv (x-2)(x^2 + x - 12) \equiv (x-2)(x+4)(x-3)$

Factorise and solve

$$3x^3 + 34x^2 + 41x + 10 = 0$$

First, let $f(x) = 3x^3 + 34x^2 + 41x + 10$

Where to begin? Notice that 10 potentially has the factors ± 1, ± 2, ± 5 and ± 10.

Evaluate: $f(1) = 88$. Therefore $x - 1$ is not a factor of $f(x)$.

Evaluate: $f(-1) = 0$. Therefore $x + 1$ is a factor of $f(x)$.

Let's use polynomial division to split this:

$$\begin{array}{r l}
 & \quad 3x^2 + 31x + 10 \\ \cline{2-2}
x+1 \big) & \quad 3x^3 + 34x^2 + 41x + 10 \\
 & -3x^3 \;\; -3x^2 \\ \cline{2-2}
 & \qquad 31x^2 + 41x \\
 & \quad -31x^2 - 31x \\ \cline{2-2}
 & \qquad\qquad 10x + 10 \\
 & \qquad\qquad -10x - 10 \\ \cline{2-2}
 & \qquad\qquad\qquad 0
\end{array}$$

The polynomial division tells us that

$$\frac{3x^3+34x^2+41x+10}{x+1} \equiv 3x^2+31x+10$$

Rearrange to get:

$$3x^3+34x^2+41x+10 \equiv (x+1)(3x^2+31x+10)$$

Can we factorise the quadratic? Yes!

$$3x^3+34x^2+41x+10 \equiv (x+1)(x+10)(3x+1)$$

Therefore

$$\begin{aligned} 3x^3+34x^2+41x+10 &= 0 \\ (x+1)(x+10)(3x+1) &= 0 \end{aligned}$$

Which gives $x=-1$, $x=-10$ and $x=-\frac{1}{3}$.

Remainder theorem

Remainder theorem

When a polynomial $f(x)$ is divided by a linear term $x-a$ then the remainder is given by $f(a)$.

Remainder theorem gives the remainder without carrying out polynomial division. It is useful when the quotient is not required.

It has a simple proof: for polynomial division with a linear divisor then

$$\frac{f(x)}{x-a} \equiv q(x) + \frac{r}{x-a}$$

where $q(x)$ is the quotient and r is the remainder. If $x-a$ is not factor of $f(x)$ then the remainder will be non-zero.

Multiply both sides by $x-a$ to obtain:

$$f(x) = (x-a)q(x) + r$$

When $x = a$ we have the obvious result:

$$f(a) = r$$

So we can find the remainder r by evaluating $f(a)$.

Remainder theorem

Find the remainder after division for:

$$\frac{x^3+7x-5}{x+2}$$

Let $f(x)=x^3+7x-5$.

Evaluate $f(-2)=-8-14-5=-27$.

The remainder after division is -27.

$$
\begin{array}{r|rrrr}
 & x^2 & -2x & +11 & \\
\hline
x+2 & x^3 & & +7x & -5 \\
 & -x^3 & -2x^2 & & \\
\hline
 & & -2x^2 & +7x & \\
 & & 2x^2 & +4x & \\
\hline
 & & & 11x & -5 \\
 & & & -11x & -22 \\
\hline
 & & & & -27
\end{array}
$$

Test yourself...

You should be able to solve the following problems if you have understood everything in these notes.

1. Show that $2x+3$ is a factor of $2x^3+3x^2-2x-3$.
2. Find the remainder of $\dfrac{x^5+2x-5}{x+3}$.
3. Fully factorise $2x^3+7x^2-17x-10$.

Answers:

1. $f(x)=2x^3+3x^2-2x-3$. Since $f(-\frac{3}{2})=0$ then $2x+3$ is a factor.
2. $f(x)=x^5+2x-5$. Then remainder is $f(-3)=-254$.
3. $(x-2)(x+5)(2x+1)$