Polynomial Division

Algebra 3

INU0114/514 (Maths 1)

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Overview

We will develop some useful skills and knowledge of algebra.

- Know the definition of polynomial
- Be able to carry out polynomial division to obtain a quotient and remainder.
    - Synthetic division
    - Comparing coefficients
- Be able to use polynomial division to find the equation of a tangent line to a polynomial.

Definitions

Polynomial

is an expression constructed from variables and constants. The terms of a polynomial are joined by the operations of addition, subtraction, multiplication and non-negative integer powers.

For example, $x^3 - 5x + 7$ is a polynomial.

But $x^2 - 7x^{-1} + 3x^{\frac{2}{5}}$ is not, because the second term contains a division ($x^{-1} = \frac{1}{x}$) and the third term contains a fractional index.

Degree

The degree of a polynomial is the value of the highest power (index) within the polynomial.

For example $x^3 - 3x^2 + 5x - 7$ is a polynomial of degree 3.

Polynomial division

Polynomial division (also called long division) is a method for dividing a polynomial by another of the same or smaller degree.

It can be used to transform *improper algebraic fractions* into expressions containing *proper algebraic fractions*. The result of carrying out the division will be polynomials that have lower degree than the original polynomial.

This is how we represent this process: given two polynomials $f(x)$ and $g(x) \neq 0$:

$$\frac{f(x)}{g(x)} \equiv q(x) + \frac{r(x)}{g(x)}$$

The function $q(x)$ is called the *quotient* and the function $r(x)$ is called the *remainder*. It is possible that the quotient and remainder after division will be constants or functions.

Polynomial division

Carry out the division $\dfrac{2x^2-7x-6}{x-1}$.

We will go through this in stages. The first step is to write the functions out using long division notation:

$$
\begin{array}{rrrr}
 & & 2x & -5 \\ \cline{2-4}
x-1) & 2x^2 & -7x & -6 \\
 & -2x^2 & +2x & \\ \cline{2-3}
 & & -5x & -6 \\
 & & 5x & -5 \\ \cline{3-4}
 & & & -11
\end{array}
$$

The process ends when the remainder is a degree less than the divisor.

The *quotient* is $2x-5$ and the *remainder* is -11.

Therefore the polynomial division looks like this:

$$\frac{2x^2-7x-6}{x-1} \equiv 2x-5-\frac{11}{x-1}$$

.

Polynomial division

Carry out the polynomial division $\dfrac{x^3-3x^2+x+1}{x+2}$.

Step by step...

$$
\begin{array}{r|l}
 & x^2 - 5x + 11 \\
\hline
x+2 & x^3 - 3x^2 + x + 1 \\
 & -x^3 - 2x^2 \\
\hline
 & -5x^2 + x \\
 & 5x^2 + 10x \\
\hline
 & 11x + 1 \\
 & -11x - 22 \\
\hline
 & -21
\end{array}
$$

The *quotient* is $x^2-5x+11$ and the *remainder* is -21.

Therefore the polynomial division looks like this:

$$\frac{x^3-3x^2+x+1}{x+2} \equiv x^2-5x+11-\frac{21}{x+2}$$

Polynomial division

Carry out the polynomial division $\dfrac{x^3-2x-6}{x^2+1}$.

Step by step...

$$\begin{array}{rl} & x \\ x^2+1 \big) & \overline{x^3-2x-6} \\ & \underline{-x^3\quad -x} \\ & -3x \end{array}$$

The process stops because the remainder has a degree less than the divisor. The *quotient* is x and the *remainder* is $-3x-6$.

Therefore the polynomial division looks like this:

$$\frac{x^3-2x-6}{x^2+1} \equiv x - \frac{3x+6}{x^2+1}$$

Synthetic division

- There's an alternative method you might like to use in cases where the divisor has the form $x - a$.
- It is called *synthetic division* (or sometimes as Horner's method).
- This method is generally faster (needing fewer computations and less paper) than the polynomial division already described.

We'll work through an example next.

Expand $\dfrac{x^3-6x^2-12}{x-3}$.

We layout the polynomial coefficients (using 0 for missing powers of x). The divisor is $x-3$ we set $x=3$ to the left of the bar.

$$\begin{array}{c|cccc} & 1 & -6 & 0 & -12 \\ 3 & & & & \\ \hline & & & & \end{array}$$

Drop the first coefficient straight down below the bar.

$$\begin{array}{c|cccc} & 1 & -6 & 0 & -12 \\ 3 & \downarrow & & & \\ \hline & 1 & & & \end{array}$$

Multiply the dropped coefficient by the value left of the bar and put the result in the next column.

$$\begin{array}{c|cccc} & 1 & -6 & 0 & -12 \\ 3 & & 3 & & \\ \hline & 1 \nearrow^{\cdot 3} & & & \end{array}$$

Perform an addition in the next column:

$$\begin{array}{c|cccc} & 1 & -6 & 0 & -12 \\ 3 & & 3\ + & & \\ \hline & 1 & -3 & & \end{array}$$

Repeat the multiplication and addition steps:

$$\begin{array}{c|cccc} & 1 & -6 & 0 & -12 \\ 3 & & 3 & -9 & \\ \hline & 1 & -3 \nearrow^{\cdot 3} & & \end{array}$$

$$\begin{array}{c|cccc} & 1 & -6 & 0 & -12 \\ 3 & & 3 & -9 & \\ \hline & 1 & -3 & -9 & \end{array}$$

$$\begin{array}{c|cccc} & 1 & -6 & 0 & -12 \\ 3 & & 3 & -9 & -27 \\ \hline & 1 & -3 & -9 & -39 \end{array}$$

And again:

$$\begin{array}{c|cccc} & 1 & -6 & 0 & -12 \\ 3 & & 3 & -9 & -27 \\ \hline & 1 & -3 & -9 & \end{array}$$

Last row contains the three coefficients of the quotient and the final value is the remainder:

$$\begin{array}{c|cccc} & 1 & -6 & 0 & -12 \\ 3 & & 3 & -9 & -27 \\ \hline & 1 & -3 & -9 & -39 \end{array}$$

Therefore

$$\frac{x^3-6x^2-12}{x-3} \equiv x^2-3x-9-\frac{39}{x-3}$$

Comparing coefficients

In algebraic problems we can sometimes make progress using a technique where we compare the coefficients of relevant terms.

For example, if we were presented with the identity:

$$ax^5 + 2bx^3 + cx + d \equiv 4x^5 + 6x^3 + 3x + 10$$

it is easy deduce that

$$a = 4,\ b = 3,\ c = 3 \text{ and } d = 10$$

The method of comparing coefficients is also useful in other situations, including polynomial division.

Polynomial division (comparing coefficients)

Carry out the polynomial division

$$\frac{x^3 - 3x^2 + 2x - 4}{x + 1}$$

by comparing coefficients.

The first thing we note is that after the division is carried out we'll get a quadratic quotient and probably a remainder.

The answer have the form:

$$\frac{x^3 - 3x^2 + 2x - 4}{x + 1} \equiv ax^2 + bx + c + \frac{r}{x + 1}$$

Cross multiply both sides by $x + 1$:

$$x^3 - 3x^2 + 2x - 4 \equiv (ax^2 + bx + c)(x + 1) + r$$

Expand the RHS and group like-terms:

$$\begin{aligned} x^3 - 3x^2 + 2x - 4 &\equiv ax^3 + ax^2 + bx^2 + bx + cx + c + r \\ &\equiv ax^3 + (a + b)x^2 + (b + c)x + c + r \end{aligned}$$

We saw that

$$x^3 - 3x^2 + 2x - 4 \equiv ax^3 + (a+b)x^2 + (b+c)x + c + r$$

Compare powers of x^3 on both sides: $1x^3 = ax^3$ so $a = 1$.

For the x^2 terms: $-3 = a + b$ so $b = -4$.

For the x terms: $2 = b + c$ so $c = 6$.

And the constant terms: $-4 = c + r$ so that $r = -10$.

Therefore

$$\frac{x^3 - 3x^2 + 2x - 4}{x+1} \equiv x^2 - 4x + 6 - \frac{10}{x+1}$$

Tangent to a curve

Polynomial division will have practical applications when see integral calculus much later in the course.

For now let's see how polynomial division can be used to find the equation of a tangent to a polynomial.

If $r(x)$ is the remainder of the division of $f(x)$ by $(x-a)^2$ then the equation of the tangent line at $x=a$ to the graph of the function $y=f(x)$ is $y=r(x)$, regardless of whether or not a is a root of the polynomial.

Let's see how this works...

Equation of a tangent

Find the equation of the tangent line to the curve

$$y = x^3 + x^2 - 4x - 6$$

at the point where $x = 2$.

We're given $x = 2$ so we use the term $(x-2)$ and carry out polynomial division by $(x-2)^2$. Now $(x-2)^2 \equiv x^2 - 4x + 4$ so

$$\begin{array}{r|l}
 & x + 5 \\
x^2 - 4x + 4 & x^3 + x^2 - 4x - 6 \\
 & -x^3 + 4x^2 - 4x \\
 & \quad 5x^2 - 8x - 6 \\
 & \quad -5x^2 + 20x - 20 \\
 & \qquad 12x - 26
\end{array}$$

The tangent equation is $y = 12x - 26$.

Equation of a tangent

Find the equation of the tangent line to the curve

$$y = x^3 - 3x + 2$$

at the point where $x = -2$.

We're given $x = -2$ so we use the term $(x + 2)$ and carry out polynomial division by $(x + 2)^2$.

Now $(x + 2)^2 \equiv x^2 + 4x + 4$ so

$$
\begin{array}{r|rrrr}
 & & x & -4 & \\
\hline
x^2+4x+4 & x^3 & & -3x & +2 \\
 & -x^3 & -4x^2 & -4x & \\
\hline
 & & -4x^2 & -7x & +2 \\
 & & 4x^2 & +16x & +16 \\
\hline
 & & & 9x & +18
\end{array}
$$

The tangent equation is $y = 9x + 18$.

Test yourself...

You should be able to solve the following problems if you have understood everything in these notes.

1. Expand the expression $\dfrac{x^2-5x+14}{x-6}$
2. Simplify $\dfrac{3x^3+5x^2+x-1}{3x-1}$
3. Use polynomial division to find the tangent equation to the curve

$$y=2x^3-x^2-3x-1$$

at the point where $x=1$.

Answers:

1. $x+1+\dfrac{20}{x-6}$
2. x^2+2x+1
3. $y=x-4$