Quadratic Graphs

Algebra 2

INU0114/514 (Maths 1)

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Objectives

- Be able to sketch the graph of a quadratic function
  - Recognise the shape of the parabola from the function.
  - Use method “completing the square” to find turning point and lines of symmetry.
  - Find intersection points with the coordinate axes.
- Understand why “sketching” is better than “plotting”.

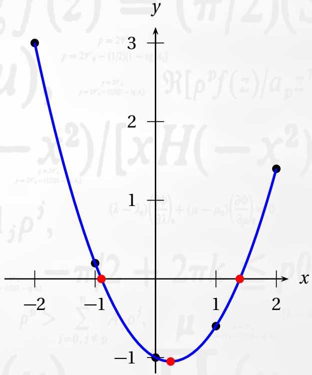
In your previous education it is likely that you were asked to plot the graph of a function - perhaps a straight line, a quadratic or something else.

Plotting a quadratic

Plot the graph of $y = 0.8x^2 - 0.4x - 1$.

Make a table of values for x and y.

x	−2	−1	0	1	2
y	3	0.2	−1	−0.6	1.4



Plot the points on a graph. Then draw a smooth curve through the points.
An advantage of this method is that it's easy.
However: important features (the x intersections and the minimum point) are missed because we only plotted certain x values.

Sketching quadratic functions

Given the general quadratic function

$$y = ax^2 + bx + c$$

The graph of the function can be sketched by considering the following:

1. The shape of the curve.
2. The point where the curve intersects with the y-axis.
3. Does the curve intersect with the x-axis. If so, what are the coordinates of the points of intersection.
4. The location of the vertex (turning point) of the curve.

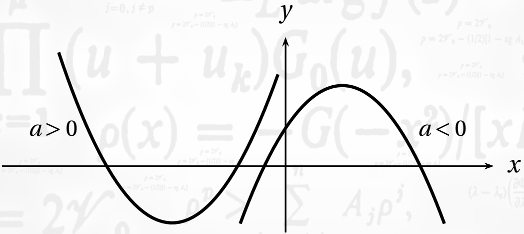
Sketching quadratic functions

Shape of the curve

Given the general quadratic function

$$y = ax^2 + bx + c$$

There are two possible shapes for the graph of the quadratic curve.



- If $a > 0$ then the quadratic curve is u-shaped or *concave up*.
- If $a < 0$ then the quadratic curve is n-shaped or *concave down*.

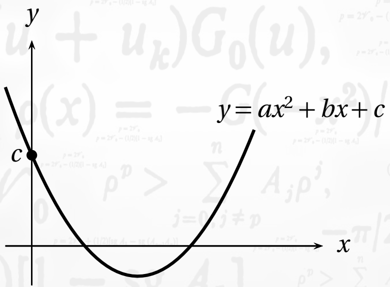
Sketching quadratic functions

Intercept with the y-axis

Given the general quadratic function

$$y = ax^2 + bx + c$$

When $x = 0$ then $y = c$.



That means the curve crosses the y-axis at the point $(0, c)$.

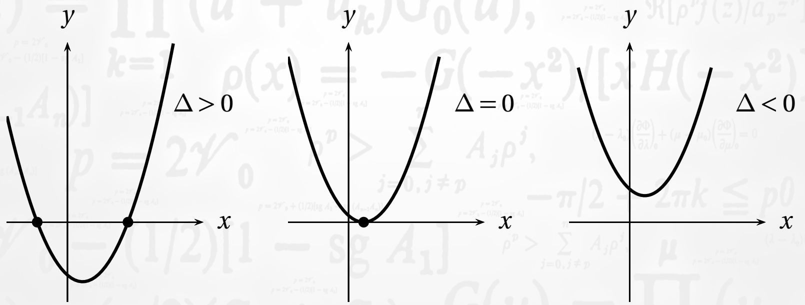
Sketching quadratic functions

Intercept with the y-axis

Given the general quadratic function

$$y = ax^2 + bx + c$$

Provided the discriminant is nonnegative $\Delta \geq 0$, then we can calculate the roots of the equation $ax^2 + bx + c = 0$ to find intersections with the x-axis.



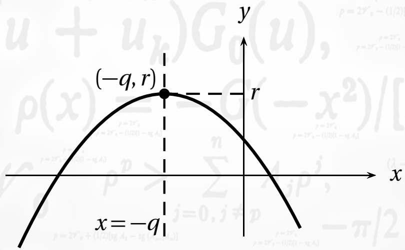
Sketching quadratic functions

Turning point and line of symmetry

We'll use the method of completing the square to help sketch quadratics.

$$y = ax^2 + bx + c \xrightarrow{\text{"complete the square"}} y = p(x+q)^2 + r$$

where p, q and r are real numbers.



The parabola has a line of symmetry at $x = -q$.

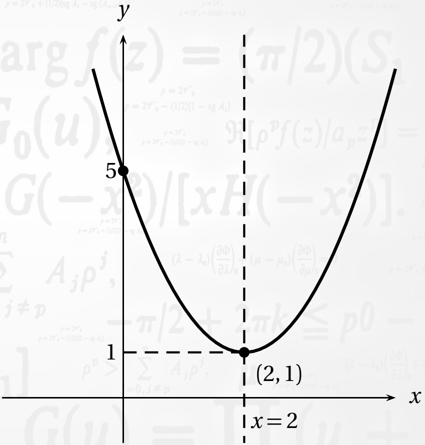
The turning point of the parabola has coordinates $(-q, r)$.

Sketching a quadratic

Sketch the graph of $y=(x-2)^2+1$

This quadratic function is already in the useful “completed square” form.

- Expanding the brackets would give a positive x^2 term; the curve is u-shaped.
- Line of symmetry at $x=2$.
- Turning point (minimum) at $(2,1)$. (The curve is completely above the x-axis)
- Expand the brackets and simplify: $y=x^2-4x+5$. The y-intercept is at $(0,5)$.



Sketching a quadratic

Sketch the graph of $y = x^2 - x - 6$

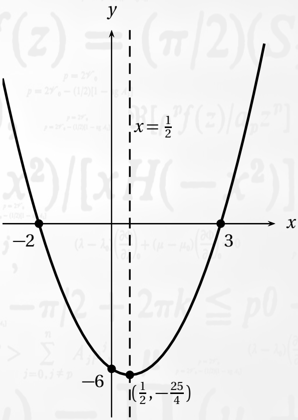
- Positive x^2 term; the curve is u-shaped.
- The y-intercept at $(0, -6)$.
- Factorise

$$y = (x+2)(x-3)$$

- Intersections at $x = 3$, $x = -2$.
- Complete the square:

$$y = (x - \tfrac{1}{2})^2 - \tfrac{25}{4}$$

- Line of symmetry: $x = \frac{1}{2}$
- Minimum point: $(\frac{1}{2}, -\frac{25}{4})$.



Sketching a quadratic

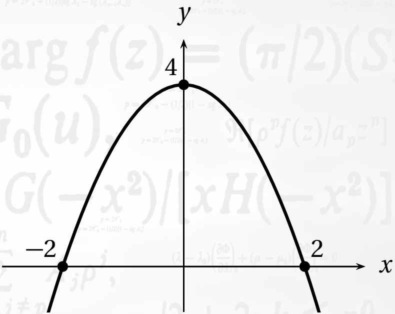
Sketch the graph of $y = 4 - x^2$

- We have a negative x^2 term; the curve is n-shaped.
- The y-intercept is at $(0,4)$.

Factorise the function:

$$y = (2-x)(2+x)$$

- Intersection points with the x-axis at $(2,0)$ and $(-2,0)$.
- Line of symmetry is $x = 0$
- Maximum point is at $(0,4)$



Now sketch the curve!

Sketching a quadratic

Sketch the graph of $y = 3 + 4x - 2x^2$

- We see a negative x^2 term; the curve is n-shaped.
- The y-intercept is at $(0, 3)$.

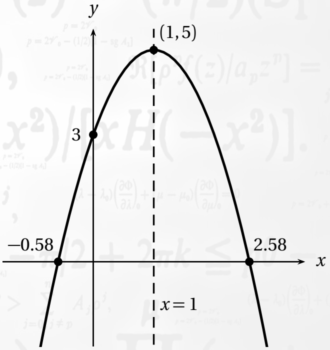
Express the function in completed square form:

$$y = -2(x-1)^2 + 5$$

- Line of symmetry at $x = 1$
- Maximum point at $(1, 5)$

Solve the equation $3 + 4x - 2x^2 = 0$ to locate the roots:

- $x = 1 \pm \frac{1}{2}\sqrt{10}...$
- $x \approx 2.6$, $x \approx -0.6$ (1 D.P.)



Sketching a quadratic

Sketch the graph of $y = 0.8x^2 - 0.4x - 1$

- We see a positive x^2 term; the curve is u-shaped.
- The y-intercept is at $(0, -1)$.

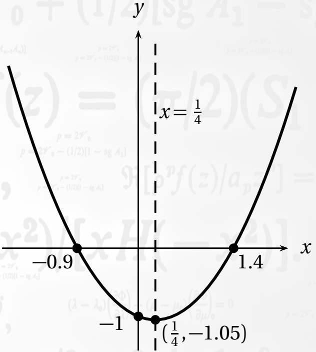
Express the function in completed square form:

$$y = 0.8(x - \tfrac{1}{4})^2 - 1.05$$

- Line of symmetry at $x = \frac{1}{4}$
- Minimum point at $(\frac{1}{4}, -1.05)$

Solve the equation $0.8x^2 - 0.4x - 1 = 0$ to locate the roots:

- $x = \frac{1}{4} \pm \frac{1}{4}\sqrt{21}...$
- $x \approx -0.9$, $x \approx 1.4$ (1 D.P.)



Compare this to how we plotted the first example: this time we found all the important points on the graph.

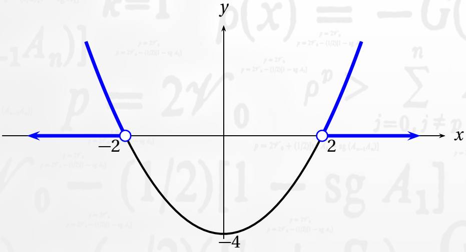
Quadratic inequalities

Although there are algebraic methods for solving inequalities, it is often quicker to visualise the solution using the quadratic graph.

Consider the inequality

$$x^2 - 4 > 0$$

Here is the corresponding graph.



Solutions to the inequality are those places where the curve is above the x-axis. It is easy to see that the solutions must be:

$$x < -2 \quad \text{and} \quad x > 2$$

Discriminant and roots

Consider the quadratic equation

$$x^2 - px + 2p = 0$$

Find the value of p for which the equation has complex roots.

In this case we have $a = 1$, $b = -p$ and $c = 2p$.

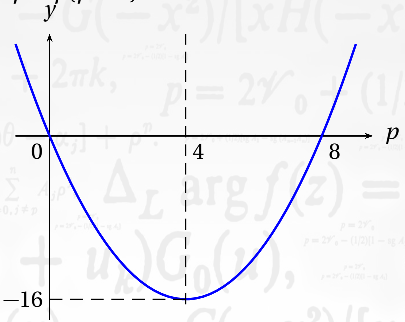
The condition on the coefficients to give complex roots is $b^2 - 4ac < 0$.

Therefore

$$\begin{aligned} (-p)^2 - 4(1)(2p) &< 0 \\ p^2 - 8p &< 0 \\ p(p-8) &< 0 \end{aligned}$$

Now we must solve the inequality to find the values of p. This is quickly done using a sketch.

The graph of $y = p^2 - 8p = p(p - 8)$ is shown below.



To solve the inequality $p^2 - 8p < 0$ we look that part of the curve which is less than zero — below the axis.

Clearly, this happens when p is between 0 and 8.

Therefore the solutions to

$$x^2 - px + 2p = 0$$

are complex when $0 < p < 8$.