QUADRATIC DISCRIMINANT

ALGEBRA 2

INU0114/514 (MATHS 1)

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Objectives

- Use the discriminant to analyse the roots of a quadratic which may be
  - Real and different
  - Real and equal
  - Complex
- Form and solve inequalities related to the discriminant (linear and quadratic).

Nature of the roots: the discriminant

Consider the quadratic formula for calculating the roots:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression under the root determines the type of solutions to the quadratic. It is called the discriminant and is represented by the symbol Δ, so that

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}, \text{ where } \Delta = b^2 - 4ac$$

There are three possible outcomes for Δ depending on the values of a, b and c.

- $\Delta > 0$ will give real and distinct (different) roots.
- $\Delta = 0$ will give one value of x, meaning two real and equal roots.
- $\Delta < 0$ will involve the square-root of a negative number, meaning there are two complex roots.

Using the discriminant

Classify the roots of the equation $4x^2 + 8x + 3 = 0$.

Here we have $a = 4$, $b = 8$, $c = 3$. The discriminant is

$$\Delta = 8^2 - (4)(4)(3) = 64 - 48 = 16$$

Since $\Delta > 0$ then this equation has two real and distinct roots.

Using the discriminant

What type of roots does the equation $2x^2 + x + 1 = 0$ have?

Here we have $a = 2$, $b = 1$, $c = 1$. The discriminant is

$$\Delta = 1^2 - (4)(2)(1) = 1 - 8 = -7$$

Since $\Delta < 0$ then this equation has two complex roots.

Using the discriminant

Describe the nature of the roots of $x^2 - 10x + 25 = 0$.

Here we have $a = 1$, $b = -10$, $c = 25$. The discriminant is

$$\Delta = (-10)^2 - (4)(1)(25) = 100 - 100 = 0$$

Since $\Delta = 0$ then this equation has two real and equal roots.

Quadratic roots

Consider the quadratic equation

$$2x^2 - 4x + k = 0$$

Find the value of k for which the equation has real, equal roots.

In this case we have $a = 2$, $b = -4$ and $c = k$.

The condition on the coefficients to give real and equal roots is $b^2 - 4ac = 0$.

Therefore

$$\begin{aligned} (-4)^2 - 4(2)k &= 0 \\ 16 - 8k &= 0 \\ 8k &= 16 \\ k &= 2 \end{aligned}$$

The quadratic will have real, equal roots when $2x^2 - 4x + 2 = 0$.

Discriminant (and quadratic inequalities)

Discriminant: real roots

Consider the quadratic equation

$$x^2 + 2kx + 1 = 0$$

Find the value of k for which the equation has real roots.

In this case we have $a = 1$, $b = 2k$ and $c = 1$.

The condition on the coefficients to give real roots is $b^2 - 4ac \geq 0$.

Therefore

$$\begin{aligned} (2k)^2 - 4(1)(1) &\geq 0 \\ 4k^2 - 4 &\geq 0 \\ k^2 - 1 &\geq 0 \end{aligned}$$

Factorise to get

$$(k-1)(k+1) \geq 0$$

How do we solve this?

Consider the inequality $(k-1)(k+1) \geq 0$.

Let's consider some k-values for each term on the LHS and look for when the product is positive or zero:

k	-3	-2	-1	0	1	2	3
$k-1$	-4	-3	-2	-1	0	1	2
$k+1$	-2	-1	0	1	2	3	4
$(k-1)(k+1)$	$+$	$+$	0	$-$	0	$+$	$+$

The results suggest that $(k-1)(k+1)$ is positive for all values except those k-values between -1 and 1.

Therefore the equation

$$x^2 + 2kx + 1 = 0$$

has real roots when $k \leq -1$ and $k \geq 1$.

Quadratic inequalities are more difficult to solve than the linear inequalities considered earlier in the course

Arguably, an easier way to solve them is by considering the graph of the quadratic (in this case $y = k^2 - 1$) and using that to identify solutions. More about this in the next presentation.

Test yourself...

You should be able to solve the following problems based on the material covered so far.

1. Classify the roots of $x^2 + 4x + 1 = 0$
2. Classify the roots of $-2 - 12x = 20x^2$
3. Classify the roots of $36x^2 + 84x + 49 = 0$
4. Find the value of p for which $2px^2 - 4x + p = 0$ has real, equal roots.
5. Find the value of k for which $4x^2 + kx + k = 0$ has complex roots.

Answers:

1. $\Delta = 12$; Real and distinct.
2. $\Delta = -16$; Complex.
3. $\Delta = 0$; Real and equal.
4. $p = \sqrt{2}$ and $p = -\sqrt{2}$.
5. $0 < k < 16$