Quadratic equations

Algebra 2

INU0114/514 (Maths 1)

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Objectives

The main objectives of this presentation is to develop the following key skills:

- Be able to factorise quadratic expressions
- Write a quadratic expression in a ’completed square’ form
- Solve quadratic equations
  - Factorising
  - Completing the square method
  - By formula

Overview

Quadratic equations have the form

$$ax^2 + bx + c = 0$$

where a, b and c are numbers and $a \neq 0$. The goal is to find the values of x which satisfy this equation.

Quadratic equations and arise in many areas of science and engineering. For instance:

- The position of an object moving under constant acceleration (like a projectile moving under the action of gravity).
- The relationship between pressure and speed of flowing air is described by a quadratic function.
- Complicated equations like $2\sin^2 x + 3\cos x - 7 = 0$ can be transformed to quadratic equations to find solutions.
- The *complex* solutions of a quadratic equations describe oscillating systems (harmonic oscillators, waves, vibrations and so on).

There are many, many others.

Factorising quadratic expressions ($a = 1$)

A quadratic expression of the form $x^2 + bx + c$ can be sometimes be factorised easily using the following method (known as Viete's method).

Factorising a quadratic

Factorise the quadratic expression $x^2 - 13x + 42$.

To factorise this expression we look for factors of 42 with sums of -13.

Possible factors of 42 are (1,42) , (2,21) , (3, 14) , (6, 7), (-1,-42) , (-2,-21) , (-3, -14) , (-6, -7).

The only factors which also have a sum of -13 are (-6,-7).

Therefore $(x-6)$ and $(x-7)$ are the required factors.

$$x^2 - 13x + 42 \equiv (x-6)(x-7)$$

Solution by factorising

If a quadratic equation $x^2 + bx + c = 0$ can be factorised to give

$$(x-\alpha)(x-\beta) = 0$$

then the solutions to the equation are $x = \alpha$ and $x = \beta$.

Solving a quadratic by factorising

Solve the quadratic equation $x^2 + x - 110 = 0$.

Look for values with a sum of 1 and a product of -110.

(The values 11 and -10 satisfy this condition!)

Therefore the LHS (left-hand side) can be written

$$(x-10)(x+11) = 0$$

So the solutions are $x = 10$ and $x = -11$.

Solution by inspection

Easy quadratic equation ($b = 0$)

Solve the equation $2x^2 - 18 = 0$

In this case we just rearrange the equation and use the square-root.

$$\begin{aligned} 2x^2 &= 18 \\ x^2 &= 9 \\ x &= \pm 3 \end{aligned}$$

We take the positive and negative roots because they both satisfy the original equation.

Easy quadratic equation ($c = 0$)

Solve the equation $5x^2 - 20x = 0$

The LHS of the equation is easily factorised:

$$5x(x - 4) = 0$$

Therefore the solutions are $x = 0$ and $x = 4$.

Inspection relies on our experience (or skill) at recognising what to do next.

Test yourself...

You should be able to solve the following problems based on the material covered so far.

1. Factorise $x^2+14x+45$
2. Factorise $x^2-23x+60$
3. Factorise x^2-64.
4. Factorise and solve $x^2-x-30=0$.
5. Factorise and solve $x^2+5x=0$.
6. Factorise $100x^2-19$.

Answers:

1. $(x+9)(x+5)$
2. $(x-3)(x-20)$
3. $(x-8)(x+8)$
4. $x=6$, $x=-5$
5. $x=0$, $x=-5$
6. $(10x-\sqrt{19})(10x+\sqrt{19})$

Factorising a quadratic

Factorise the quadratic expression $4x^2 + 4x - 15$.

This time we look at the coefficients $a \times c$, which in this case is -60.

To factorise this expression we look for factors of -60 with a sum of 4.

The numbers 10 and -6 satsify that condition.

Split the $4x$ term apart:

$$4x^2 + 10x - 6x - 15$$

Factor the first two terms and final two terms separately:

$$2x(2x+5) - 3(2x+5)$$

Notice the common factor $2x+5$:

$$(2x+5)(2x-3)$$

Therefore $4x^2 + 4x - 15 \equiv (2x+5)(2x-3)$.

Solution by factorising

Solving a quadratic by factorising

Solve the quadratic equation $4x^2 - 25x - 21 = 0$.

Look for values with a sum of -25 and a product of $4 \times (-21) = -84$.

(The values 3 and -28 satisfy this condition)

Therefore the LHS (left-hand side) can be written

$$4x^2 - 28x + 3x - 21 = 0$$

Factorise the first and last two terms separately:

$$\begin{aligned} 4x(x-7) + 3(x-7) &= 0 \\ (x-7)(4x+3) &= 0 \end{aligned}$$

Therefore $x - 7 = 0$ and $4x + 3 = 0$.

The solutions are $x = 7$ and $x = -\frac{3}{4}$.

Test yourself...

You should be able to solve the following problems based on the material covered so far.

1. Factorise $2x^2+9x-5$
2. Factorise $4x^2-9x+2$
3. Factorise $3x^2+2x-40$.
4. Factorise and solve $4x^2-20x+25=0$.
5. Factorise and solve $10x^2+17x-6=0$.

Answers:

1. $(2x-1)(x+5)$
2. $(4x-1)(x-2)$
3. $(3x-10)(x+4)$
4. $x=\frac{5}{2}$
5. $x=-2$, $x=\frac{3}{10}$

Completing the square

Consider the following patterns obtained by squaring:

$$\begin{aligned}(x+1)^2 &\equiv x^2+2x+1\\(x+2)^2 &\equiv x^2+4x+4\\(x+3)^2 &\equiv x^2+6x+9\end{aligned}$$

...and so on. In general:

$$(x+p)^2 \equiv x^2+2px+p^2 \quad \Rightarrow x^2+2px \equiv (x+p)^2-p^2$$

Now, if we had to solve the quadratic equation $x^2+2x-15=0$.

Rearrange slightly to get:

$$x^2+2x=15$$

We could replace the LHS with

$$(x+1)^2-1=15$$

So that

$$(x+1)^2=16$$

Now take square roots of both sides:

$$x+1=\pm 4 \quad x=-1\pm 4$$

Therefore $x=-5$ and $x=3$.

Completing the square method

Given the quadratic equation

$$x^2 + 6x - 3 = 0$$

Express this equation in the form $(x+p)^2 + q = 0$, where p and q are integers, then solve it.

First we need to 'complete the square' on the RHS.

Consider the terms $x^2 + 6x$. We can replace them with $(x+3)^2 - 9$

$$\begin{aligned} x^2 + 6x - 3 &= 0 \\ (x+3)^2 - 9 - 3 &= 0 \\ \therefore (x+3)^2 - 12 &= 0 \end{aligned}$$

We solve by rearranging and making x the subject:

$$\begin{aligned} (x+3)^2 &= 12 \\ x+3 &= x \pm \sqrt{12} \qquad \therefore x = -3 \pm \sqrt{12} \end{aligned}$$

The two solutions are $x = -3 - \sqrt{12}$ and $x = -3 + \sqrt{12}$.

(We can simplify the surd or give the answers as decimals if required).

Completed square form

Give the quadratic expression

$$3x^2 - 12x + 20$$

in the form $p(x+q)^2 + r$ where p, q and r are integers.

First take a common factor of 3 from the first two terms:

$$3(x^2 - 4x) + 20$$

Now consider the $x^2 - 4x$ terms; we will replace this with $(x-2)^2 - 4$:

$$3\left[(x-2)^2 - 4\right] + 20$$

Expand the brackets

$$3(x-2)^2 - 12 + 20$$

Finally, simplify:

$$3(x-2)^2 + 8$$

Test yourself...

You should be able to solve the following problems based on the material covered so far.

In the following cases p, q and r are real numbers.

1. Express $x^2 + 8x$ in the form $(x+p)^2 + q$
2. Express $x^2 - 3x$ in the form $(x+p)^2 + q$
3. Express $x^2 + 20x + 30$ in the form $(x+p)^2 + q$
4. Express $x^2 - 11x + \frac{5}{4}$ in the form $(x+p)^2 + q$
5. Express $3x^2 - 4x - 2$ in the form $r(x+p)^2 + q$.
6. Express $30 - 6x^2 + 18x$ in the form $r(x+p)^2 + q$.

Answers:

1. $(x+4)^2 - 16$
2. $(x - \frac{3}{2})^2 - \frac{9}{4}$
3. $(x+10)^2 - 70$
4. $(x - \frac{11}{2})^2 - 29$
5. $3(x - \frac{2}{3})^2 - \frac{10}{3}$
6. $-6(x - \frac{3}{2})^2 + \frac{87}{2}$

Quadratic formula

We can derive a formula for solving a quadratic equation using the method of *completing the square.*

Starting with the general equation

$$ax^2+bx+c=0$$

Divide both sides by a:

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

Rearrange:

$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

Complete the square on the LHS

$$(x+\frac{b}{2a})^2-(\frac{b}{2a})^2=-\frac{c}{a}$$

Rearrange and put the RHS over a common denominator:

$$\begin{aligned}(x+\tfrac{b}{2a})^2 &= (\tfrac{b}{2a})^2-\tfrac{c}{a}\\ (x+\tfrac{b}{2a})^2 &= \tfrac{b^2}{4a^2}-\tfrac{c}{a}\\ (x+\tfrac{b}{2a})^2 &= \tfrac{b^2-4ac}{4a^2}\end{aligned}$$

Take square-roots:

$$\begin{aligned}x+\tfrac{b}{2a} &= \pm\sqrt{\tfrac{b^2-4ac}{4a^2}}\\ &= \pm\frac{\sqrt{b^2-4ac}}{2a}\end{aligned}$$

Rearrange for x

$$x=\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

Over a common denominator:

$$\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

To calculate the roots, we just substitute the coefficients into this formula.

Using the quadratic formula

Solve the equation $5x^2 - 17x + 6 = 0$.

For this quadratic equation we have

$$a = 5,\ b = -17,\ c = 6$$

Using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute the given values:

$$x = \frac{17 \pm \sqrt{(-17)^2 - 4(5)(6)}}{2(5)}$$

Simplify as much as possible

$$x = \frac{17 \pm \sqrt{169}}{10} = \frac{17 \pm 13}{10}$$

The solutions are $x = 3$ and $x = \frac{2}{5}$.

Test yourself...

You should be able to solve the following problems based on the material covered so far.

Solve the following equations using the quadratic formula. Give exact answers.

1. $x^2 - 17x + 66 = 0$
2. $x^2 - 47x - 150 = 0$
3. $x^2 + 2x = 40$
4. $1 - 8x - 4x^2 = 0$

Answers:

1. $x = 6$, $x = 11$
2. $x = -3$, $x = 50$
3. $x = -1 \pm \sqrt{41}$
4. $x = -1 \pm \frac{\sqrt{5}}{2}$