Logarithms

Algebra 1

INU0115/515 (Maths 2)

Dr Adrian Jannetta MIMA CMath FRAS

Logarithms and indices

Consider the mathematical statement

$$5^2 = 25$$

We can express an equivalent relationship between these numbers using the logarithm function

$$\log_5 25 = 2$$

This statement is just a different way of expressing the relationship between the numbers 2,5 and 25 in the first equation. Since they are equivalent we can write this relationship as

$$5^2 = 25 \Longleftrightarrow \log_5 25 = 2$$

The relationship between base and index is therefore

$$\boxed{a^b = c \Longleftrightarrow \log_a c = b} \quad (1)$$

It's important to note that this relationship is only completely defined when the base is positive, i.e., $a > 0$.

Index to logarithm form

To express $10^3 = 1000$ using the logarithm function,we note that the base is 10 and the logarithm (power/index) is 3. Therefore:

$$10^3 = 1000 \iff \log_{10} 1000 = 3$$

Logarithm to index form

To express $\log_7 2401 = 4$ in index form,we note that the base is 7 and the logarithm (power/index) is 4. Therefore:

$$\log_7 2401 = 4 \iff 7^4 = 2401$$

Evaluating a logarithm

In simple cases we can use the relationship between index and logarithmic form to evaluate expressions.

Index to logarithm form

Evaluate $\log_3 81$

Let

$$x = \log_3 81$$

Rewrite this relationship in index form:

$$3^x = 81$$

We can find the answer by inspection; the answer must be $x = 4$.

It is not usually practical to ‘guess’ the solution to equations so we’ll develop a better method shortly.

Types of logarithm

Although logarithms can be taken in any base only a few are commonly seen in science and engineering.

$\log_{10} x$ Logarithms in base 10 are sometimes called common logarithms. They are sometimes denoted using lg instead of $\log_{10}$.

$\log_{\mathrm{e}} x$ Logarithms in base e are usually called natural logarithms (or sometimes Naperian logarithms). They are usually written as ln rather than $\log_e$.

$\log_2 x$ Logarithms in base 2 can occur in applications related to computing, thermodynamics and statistical mechanics.

In applications where the base is not important, then the base can be omitted (this occurs when logarithms are used as a tool to solve certain types of equation).

Rules of logarithms

Logarithms are a useful tool for separating products (multiplication) and quotients (division) into addition and subtraction.

As long as all logarithms are given in the same base, we can use the following rules:

The multiply-add rule:

$$\log ab = \log a + \log b \tag{2}$$

The divide-subtract rule:

$$\log\left(\frac{a}{b}\right) = \log a - \log b \tag{3}$$

This rule is useful for expressions containing powers:

$$\log a^n = n \log a \tag{4}$$

Special results

The rules of indices can give us a few more useful results.

We know that $a^0 = 1$, therefore we can express this using a logarithm as:

$$\log_a 1 = 0 \tag{5}$$

So the logarithm of 1 in any base is zero.

Also, since $a^1 = a$ we can state that:

$$\log_a a = 1 \tag{6}$$

So, for example $\log_{10} 10 = 1$ or $\ln \mathrm{e} = 1$ (since $\ln \mathrm{e} \equiv \log_{\mathrm{e}} \mathrm{e}$).

Rules of logarithms

Here is a summary of the rules given on the previous slides.

$$\begin{aligned}\log ab &= \log a + \log b \\ \log\left(\frac{a}{b}\right) &= \log a - \log b \\ \log a^n &= n \log a\end{aligned}$$

The following are also useful:

$$\log_a 1 = 0 \qquad \log_a a = 1$$

For example $\log_7 20 + \log_7 2 = \log_7 40$.

Or $\log 50 - \log 5 = \log 10$.

Or $\log_2 20^4 = 4 \log_2 20$.

Logarithmic equations

We can solve equations containing logarithms by using equation 1 to remove the log term.

A simple logarithmic equation

Solve $\log_7 14x^2 = 3$

Express this in index form with equation (1)

$$\begin{aligned} 7^3 &= 14x^2 \\ 343 &= 14x^2 \end{aligned}$$

Rearrange and solve:

$$x^2 = \frac{343}{14} = \frac{49}{2} \quad \Rightarrow x = \pm\frac{7}{\sqrt{2}}$$

It's possible to rationalise this surd: $x = \pm\frac{7\sqrt{2}}{2}$ or evaluate to get $x = \pm 4.95$ (to 2DP).

A logarithmic equation

Solve the equation

$$\log_2(2x-5)+3=\log_2(5x-7)$$

To solve this type of equation we have to gather the log terms together on the same side of the equation. Any other terms can go on the other side.

$$\log_2(2x-5)-\log_2(5x-7)=-3$$

Use the rule $\log a-\log b=\log(a/b)$ to write:

$$\log_2\left(\frac{2x-5}{5x-7}\right)=-3$$

Remove the logarithm using the relation $a^b=c \Longleftrightarrow \log_a c=b$

$$2^{-3}=\frac{2x-5}{5x-7} \quad \Rightarrow \frac{2x-5}{5x-7}=\frac{1}{8}$$

Cross multiply both sides by $5x-7$ and rearrange to find x:

$$\begin{aligned} 2x-5 &= \frac{5x-7}{8} \\ 8(2x-5) &= 5x-7 \\ 16x-40 &= 5x-7 \\ 11x &= 33 \\ \therefore x &= 3 \end{aligned}$$

Logarithmic equations

An equation with natural logarithm

Solve $\ln x - \ln(x-1) = \frac{1}{2}$.

Use the log rule (3) to simplify the LHS:

$$\ln\left(\frac{x}{x-1}\right) = \frac{1}{2}$$

The LHS is equivalent to

$$\log_e\left(\frac{x}{x-1}\right) = \frac{1}{2}$$

So we can use equation (1) to express this in index form:

$$\frac{x}{x-1} = e^{\frac{1}{2}} = \sqrt{e}$$

Cross multiply and rearrange to get x:

$$\begin{aligned}
x &= (x-1)\sqrt{e} \\
x &= x\sqrt{e} - \sqrt{e} \\
x - x\sqrt{e} &= -\sqrt{e} \\
x(1-\sqrt{e}) &= -\sqrt{e} \\
x &= \frac{-\sqrt{e}}{1-\sqrt{e}} \\
x &= \frac{\sqrt{e}}{\sqrt{e}-1}
\end{aligned}$$

Solving exponential equations

An exponential equation is an equation where the index/exponent contains the unknown variable to be found.

A simple exponential equation

Solve the equation $7^x = 31$

Using the relationship $a^b = c \Longleftrightarrow \log_a c = b$ we can write this:

$$\log_7 31 = x$$

But what is the numerical value of x?

Some calculators are equipped to evaluate logarithms in any base. But some can only evaluate logarithms in base 10 or base e.

Next we'll see how to find the numerical value of x in this problem.

A simple exponential equation

Solve the equation $7^x = 31$

First, take logarithms of both sides:

$$\log 7^x = \log 31$$

(It does not matter at the moment whether it is base 10 or base e, the method is just the same).

Use the power rule (equation 4):

$$\begin{aligned} x\log 7 &= \log 31 \\ \therefore x &= \frac{\log 31}{\log 7} \end{aligned}$$

We can evaluate this on a calculator (in base 10 or base e) to give $x = 1.7647$ (to 4DP)

Equations with base e

If the exponential equation contains base e then we take ‘natural’ logs of both sides while solving.

Exponential equation in base e

Solve the equation $e^{2x} = 50$

First, take natural logs of both sides:

$$\ln e^{2x} = \ln 50$$

Using the rule given in (4):

$$2x \ln e = \ln 50$$

But $\ln e = \log_e e = 1$ (see earlier slides) so

$$2x = \ln 50 \quad \Rightarrow x = \frac{1}{2} \ln 50$$

We can evaluate this on a calculator: $x = 1.9560$ (to 4DP).

Using the logarithm rules

A more difficult equation

Solve the equation $5^x = 7^{x-2}$

Take logs of both sides.

$$\log 5^x = \log 7^{x-2}$$

Use the laws of logarithms

$$x \log 5 = (x-2)\log 7$$

Expand brackets on RHS:

$$x \log 5 = x \log 7 - 2\log 7$$

Get the x's on the same side:

$$x \log 7 - x \log 5 = 2\log 7$$

Factorise and rearrange to make x the subject.

$$\begin{aligned} x(\log 7 - \log 5) &= 2\log 7 \\ x &= \frac{2\log 7}{\log 7 - \log 5} \end{aligned}$$

Evaluate this to obtain an approximate answer: $x = 11.57$ to 4 significant figures.

Test yourself...

You should be able to solve the following problems if you have understood everything in these notes.

1. Express $2^{10} = 1024$ using the logarithm function.
2. Solve $3.6^{2x} = 1.5$
3. Solve $\log x^3 - \log x = \log 25$
4. Solve $e^{2x} - 6e^x + 8 = 0$. [Hint: put $y = e^x$]
5. Solve $2(3^{2x}) - 3^x - 3 = 0$.

Answers:

1. $\log_2 1024 = 10$
2. $x = \frac{\log 1.5}{2 \log 3.6}$
3. $x = 5$
4. $x = \ln 2, \ln 4$
5. $x = 1 - \frac{\log 2}{\log 3}$