SURDS

ALGEBRA 1

INU0115/515 (MATHS 2)

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Objectives

In this presentation we'll cover the following topics.

- Rational and irrational numbers
- Introduction to surds
- Arithmetic and algebra with surds
- Simplifying surds
- Rationalising surds
- What's the point of using surds?

Types of number

Rational numbers

Rational numbers can be expressed as ratios of two whole numbers (in fraction form).

Here are some rational numbers:

$$\frac{3}{4} \qquad 4=\frac{4}{1} \qquad 6.5=\frac{65}{10} \qquad 0.33333\ldots=\frac{1}{3}$$

Irrational numbers

Irrational numbers *cannot* be expressed in fraction form.

Here are some irrational numbers:

$$\sqrt{2}=1.41421\ldots \qquad \pi=3.14159\ldots \qquad \mathrm{e}=2.71828\ldots$$

Irrational numbers can't be written exactly using a decimal expansion.

Approximating irrational numbers

A famous example of an irrational number is π

$$3.14159265358979323846264\ldots$$

which represents the ratio of a circle diameter to its circumference.

In the past mathematicians have found fractions such as

$$\frac{22}{7} \text{ and } \frac{355}{113}$$

which have decimal values close to π.

But they aren't close (accurate) enough to use at this level of maths. Instead we use the symbol π in calculations and only change to a decimal at the end when all the calculations are completed.

Surds

Consider the following sequence of numbers:

$$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10$$

If we take square roots of each number:

$$\sqrt{1}\ \sqrt{2}\ \sqrt{3}\ \sqrt{4}\ \sqrt{5}\ \sqrt{6}\ \sqrt{7}\ \sqrt{8}\ \sqrt{9}\ \sqrt{10}$$

We can simplify some of numbers in this list:

$$1\ \ \sqrt{2}\ \ \sqrt{3}\ \ 2\ \ \sqrt{5}\ \ \sqrt{6}\ \ \sqrt{7}\ \ \sqrt{8}\ \ 3\ \ \sqrt{10}$$

The numbers in this list which *cannot* be simplified to an integer are called surds. Since they cannot be simplified, we leave them written using square-roots.

Motivation for using surd form

Suppose you own a calculator which can only store and display numbers to one decimal place and can do basic calculations like $+, -, \times, \div$.

Starting the $\sqrt{2}$ (which is 1.4 to one decimal place) here is what happens when we repeatedly square the number in decimal form.

$$1.4 \to 1.9 \to 3.6 \to 12.9 \to 166.4 \ldots$$

We squared the first number and then kept the first decimal place (we didn't round it). Using rules for surds (and indices) means we don't suffer so-called truncation errors in written mathematics:

$$\sqrt{2} \to 2 \to 4 \to 16 \to 256 \ldots$$

These are exact values; you can see the error in the first set of numbers gets worse and worse!

This was an extreme example; perhaps your calculator can store 10 decimal places. But same argument applies; it would just take more steps for the error to become apparent.

Using surds is a great way to avoid numerical errors in written calculations.

Arithmetic with surds

Adding and subtracting

Surds can be added and subtracted provided they are *like* terms.

$$\sqrt{5}+\sqrt{5}=2\sqrt{5} \qquad 3\sqrt{7}-8\sqrt{7}=-5\sqrt{7}$$

Unlike surds cannot be simplified: e.g. $\sqrt{3}+4\sqrt{5}$ will not simplify.

Multiplying and dividing

Surds can be multiplied or divided by numbers:

$$4(2\sqrt{3})=8\sqrt{3} \qquad \frac{12\sqrt{11}-8}{4}=3\sqrt{11}-2$$

Combining surds

Surds can be multiplied (or separated) using this rule:

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

Another sometimes useful rule:

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

These rules are consistent with the rules we already saw for indices. For example:

$$a^{\frac{1}{2}} \times b^{\frac{1}{2}} = (ab)^{\frac{1}{2}}$$

Combining surds

Simplify the expression $2\sqrt{5}\times 3\sqrt{2}$.

Multiply like terms together (surds with other surds, etc).

$$2\sqrt{5}\times 3\sqrt{2} = 2\times 3\times \sqrt{5}\times \sqrt{2} = 6\times \sqrt{5\times 2} = 6\sqrt{10}$$

Combining surds

Simplify the expression $\dfrac{2\sqrt{20}}{\sqrt{5}}$.

Combine the surds under the roots and simplify

$$\frac{2\sqrt{20}}{\sqrt{5}} = 2\sqrt{\frac{20}{5}} = 2\sqrt{4} = 2\times 2 = 4$$

Algebra with surds

The usual rules of algebra apply when we have to multiply expressions containing surds together.

Example

Expand the expression $(\sqrt{2}+3)(5-\sqrt{3})$

Multiply each term within the first set of brackets with both terms in the second:

$$(\sqrt{2}+3)(5-\sqrt{3})$$

$$\begin{aligned}
&= 5\sqrt{2}-\sqrt{3}\sqrt{2}+(3)(5)-3\sqrt{3} \\
&= 5\sqrt{2}-\sqrt{3\times 2}+15-3\sqrt{3} \\
&= 5\sqrt{2}-\sqrt{6}-3\sqrt{3}+15
\end{aligned}$$

Simplifying surds

Definition

Square numbers are the numbers obtained by squaring the integers. The first few square numbers are 1, 4, 9, 16, 25, 36, ...

Surds can be simplified by factorising the number inside the root into a square number and another integer.

Consider the surd $\sqrt{8}$.

Since we can factorise the number inside the square-root with $8 = 4 \times 2$ then the surd can be written

$$\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}$$

Changing the form of the surd from $\sqrt{8}$ to $2\sqrt{2}$ is called *simplifying*.

Simplifying a surd

Simplify the expression $\sqrt{18}+\sqrt{32}$.

The first term can be simplified as follows:

$$\sqrt{18}=\sqrt{9\times 2}=\sqrt{9}\times\sqrt{2}=3\sqrt{2}$$

The second term is simplified by writing:

$$\sqrt{32}=\sqrt{16\times 2}=\sqrt{16}\times\sqrt{2}=4\sqrt{2}$$

Therefore the original expression is

$$\begin{aligned}\sqrt{18}+\sqrt{32} &= 3\sqrt{2}+4\sqrt{2}\\ &= 7\sqrt{2}\end{aligned}$$

Modern calculators will automatically simplify expressions like these but you should still be able to simplify them for yourself!

Fractions containing surds

Fractions can contain surds, for example $\frac{\sqrt{3}}{2}$.

If the denominator (bottom) of the fraction contains a surd then we usually try to rewrite it. The process of removing a surd from the denominator (the bottom) of a fraction is called rationalising.

There are two cases we need to be able to rationalise.

They look like this:

$$\frac{1}{2\sqrt{2}} \quad \text{and} \quad \frac{3}{1+2\sqrt{2}}$$

In the first - we have one term. In the second case there are two terms - one of which contains a surd.

Note that the process of rationalising changes the surd into an equivalent form which has the same value.

Rationalising (simple case)

In the simplest case rationalising the denominator done by multiplying the top and bottom by the same surd.

Rationalising the denominator

Rationalise $\dfrac{4}{\sqrt{2}}$.

We multiply the top and bottom of the fraction by $\sqrt{2}$ and simplify if we can.

$$\frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{\sqrt{2\times 2}} = \frac{4\sqrt{2}}{\sqrt{4}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

Rationalising in this way can greatly simply the surd.

The difference of two squares

Consider what happens when we expand brackets on the expression

$$(5+\sqrt{2})(5-\sqrt{2})$$

Multiplying the terms out, we get:

$$\begin{aligned}(5+\sqrt{2})(5-\sqrt{2}) &= 25-5\sqrt{2}+5\sqrt{2}-(\sqrt{2})^2 \\ &= 25-2 \\ &= 23\end{aligned}$$

In this case the answer is a rational number — the surds cancelled out. This kind of canceling always happens when we have an expression of the form $(a+b)(a-b)$. We always get:

$$(a+b)(a-b)=a^2-b^2$$

The RHS is a difference between two square numbers and the LHS are a conjugate pair.

The *conjugate* of the expression $p+q$ is found by reversing the sign between the terms.

Rationalising (harder case)

Rationalising a surd

Rationalise the expression $\dfrac{6}{2-\sqrt{2}}$.

To get started — multiply top and bottom by the conjugate of the denominator

$$\begin{aligned}
\frac{6}{2-\sqrt{2}} &= \frac{6}{2-\sqrt{2}} \times \frac{2+\sqrt{2}}{2+\sqrt{2}} \\
&= \frac{12+6\sqrt{2}}{2^2-(\sqrt{2})^2} \\
&= \frac{12+6\sqrt{2}}{4-2} \\
&= \frac{12+6\sqrt{2}}{2} \\
\therefore \frac{6}{2-\sqrt{2}} &= 6+3\sqrt{2}
\end{aligned}$$

What's the point of this?

Before calculators were commonplace it was necessary to rationalise surds before long division could be carried out to obtain the decimal value.

This is no longer the case. Calculators are everywhere!

It is good practice to rationalise surds because it makes it more likely that we'll always end up with simpler equations to solve.

For example: consider the equation

$$x^{\frac{1}{2}} + \frac{3}{\sqrt{3}} - \sqrt{27} = 0$$

Rationalising each surd changes the equation to this:

$$x^{\frac{1}{2}} - 2\sqrt{3} = 0$$

which can be rearranged and solved more easily:

$$x = (2\sqrt{3})^2 = 12$$

A final note...

All of the surds seen have involved square roots. But surds are more general can involve cube roots $\sqrt[3]{}$ or fourth roots $\sqrt[4]{}$ or...n^{th} roots. For example:

$$\sqrt[3]{2} \quad \frac{1}{\sqrt[4]{5}} \quad \frac{1+\sqrt[6]{100}}{\sqrt{5}}$$

The principles seen in this presentation can be extended to those cases too.

For example, to simplify $\sqrt[3]{54}$ we can find cube-root factors of 54:

$$\sqrt[3]{54} = \sqrt[3]{27 \times 2} = \sqrt[3]{27} \times \sqrt[3]{2} = 3\sqrt[3]{2}$$

Test yourself...

You should be able to solve the following problems if you have understood everything in these notes.

1. Simplify $8\sqrt{3}-\sqrt{3}$.
2. Express $\sqrt{28}$ in the simplest possible form.
3. Expand $5\sqrt{3}(2-\sqrt{3})$
4. Express $\dfrac{3}{\sqrt{6}}$ as a fraction with a rational denominator.
5. Rationalise the expression $\dfrac{4}{2-\sqrt{3}}$

Answers:

1. $7\sqrt{3}$
2. $2\sqrt{7}$
3. $10\sqrt{3}-15$
4. $\frac{\sqrt{6}}{2}$
5. $8+4\sqrt{3}$