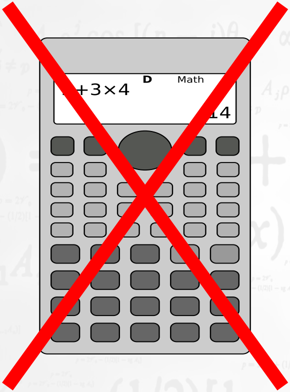
Rules of Indices

Algebra 1

INU0115/515 (Maths 2)

Dr Adrian Jannetta MIMA CMath FRAS

Objectives



- In this presentation we'll learn the rules for manipulating expressions with powers (indices or exponents).
- Understanding how these rules work will help us to solve equations containing terms with powers (e.g. $x^3 = 64$).
- The emphasis this week will be on not using a calculator.
- We'll try to develop our mental ability to apply the rules wherever possible!

Definitions

Consider the following product:

$$5 \times 5 \times 5 \times 5 = 625$$

The number 5 has been multiplied with itself 4 times.We can write this as:

$$5^4 = 625$$

In this case the number 4, written as a small “superscript” is called the index.

Index and base

When a number c is given using index notation as

$$a^b = c$$

The number a is called the *base* and b is the *index.* The index is also commonly called the *power* or *exponent.*

Speaking the maths

When the index in a terms is equal to 2 (for example x^2) we say the term is being *squared.*

When the index is equal to 3 (e.g. 5^3) then we say the term is being *cubed.*

It is usual to refer to powers of 2 or 3 in this way. Higher powers (4,5,6 ...) do not have equivalent words.

When speaking these kinds of expressions aloud, you could do it as follows:

- 3^7 is spoken “three to the power seven”.
- 10^2 is spoken “10 squared” *or* “10 to the power 2”
- 8^3 is spoken “8 cubed” *or* “8 to the power 3”
- $x^{\frac{1}{2}}$ is spoken “x to the power of a half”.

Multiplication rule

Consider the product $3^4 \times 3^2$

This actually represents the product $(3 \times 3 \times 3 \times 3) \times (3 \times 3)$. Since there are six 3's in total we can say that

$$3^4 \times 3^2 = 3^{4+2} = 3^6$$

The new index is simply the sum of the original indices.The general result when multiplying numbers or variables with indices is:

$$\boxed{a^m \times a^n = a^{m+n}} \tag{1}$$

The important point to note is that the rule only works when the value of the base a is common to both.

Simplifying indices

Simplify the expression $2^8 \times 2^3 \times 2^{-5}$ to the form 2^k.

Just add the powers:

$$2^8 \times 2^3 \times 2^{-5} = 2^{8+3-5} = 2^6$$

Simplifying indices

Simplify the expression $10x^3y^2 \times 8x^6y$

Multiply the corresponding variables together:

$$\begin{aligned} 10x^3y^2 \times 8x^6y &= 10 \times 8 \times (x^3)(x^6) \times (y^2)(y) \\ &= 80 \times x^{3+6} \times y^{2+1} \\ &= 80x^9y^3 \end{aligned}$$

Division rule

Consider the quotient $3^7 \div 3^5$

We can express this ratio as follows:

$$\begin{aligned} \frac{3^7}{3^5} &= \frac{\not{3} \times \not{3} \times \not{3} \times \not{3} \times \not{3} \times 3 \times 3}{\not{3} \times \not{3} \times \not{3} \times \not{3} \times \not{3}} \\ &= 3 \times 3 \\ \frac{3^7}{3^5} &= 3^2 \end{aligned}$$

Also, we observe that $7-5=2$ so we could write $3^7 \div 3^5 = 3^{7-5} = 3^2$.

The new index is found by the subtracting the indices in the order they appear (from left to right). The general rule for division is therefore:

$$\boxed{a^m \div a^n = a^{m-n}} \qquad (2)$$

Power rule

Terms containing indices can themselves be raised to a power.

Consider the expression $(5^2)^3$

The outer index, 3, means we have to multiply 5^2 with itself 3 times. We could write this out in full as

$$5^2 \times 5^2 \times 5^2 = (5 \times 5) \times (5 \times 5) \times (5 \times 5) = 5^6$$

In this case the final index is the product of the original indices:

$$(5^2)^3 = 5^{2\times 3} = 5^6$$

The general rule for raising numbers with indices to a new power is:

$$\boxed{(a^m)^n = a^{mn}} \qquad (3)$$

Power rule

Simplify $(3x^2)^4$

The power of four is applied to both the “3” and the x^2 term:

$$(3x^2)^4 = 3^4(x^2)^4$$

Obviously 3^4 is 81. And we use equation 3 to simplify the powers of x:

$$(3x^2)^4 = 81(x^{2\times4}) = 81x^8$$

Negative indices

Consider the following series of obvious results:

Reduce index by 1 ↓

$$\begin{aligned} 4^4 &= 4\times4\times4\times4 \\ 4^3 &= 4\times4\times4 \\ 4^2 &= 4\times4 \\ 4^1 &= 4 \end{aligned}$$

Divide by 4

Continuing the pattern, we obtain the following expressions:

$$\begin{aligned} 4^0 &= 1 \\ 4^{-1} &= \tfrac{1}{4} \\ 4^{-2} &= \tfrac{1}{4\times4} = \tfrac{1}{4^2} \\ 4^{-3} &= \tfrac{1}{4\times4\times4} = \tfrac{1}{4^3} \end{aligned}$$

The general result for any base is:

$$\boxed{a^{-n} = \frac{1}{a^n}} \qquad (4)$$

Other useful results

On the previous slide we saw that $4^1 = 4$. The general result, which is always true:

$$\boxed{a^1 = a}$$

Now consider the quotient $a^m \div a^m$ which is clearly equal to 1. Also, using the division rule for indices get:

$$a^m \div a^m = a^{m-m} = a^0$$

Therefore

$$\boxed{a^0 = 1}$$

Any number raised to the power zero is equal to one.

Roots and indices

We can write square-roots using indices. A square-root $\sqrt{a}$ is a number, which when multiplied by itself gives the original number a. Let

$$\sqrt{a} = a^k$$

Square both sides:

$$\begin{aligned} a &= (a^k)^2 \\ &= a^{2k} \end{aligned}$$

We can write the LHS as a^1 so

$$a^1 = a^{2k}$$

Since the bases are equal this means that $2k = 1$, which we solve to get $k = \frac{1}{2}$. Therefore

$$\boxed{\sqrt{a} = a^{\frac{1}{2}}}$$

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Fraction indices

The nth root of the number a is a number which, when multiplied by itself n times, is equal to the original number. It can be shown that this general result is true:

$$\boxed{\sqrt[n]{a} = a^{\frac{1}{n}}}$$

For example, $\sqrt[3]{x} = x^{\frac{1}{3}}$. Combine this result with the "power rule" (equation (3)) to get:

$$\boxed{\sqrt[n]{a^m} = (a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}}$$

Evaluating fractional indices

Evaluate $27^{\frac{2}{3}}$.

We can separate the fraction into two different powers and simplify:

$$27^{\frac{2}{3}} = (27^{\frac{1}{3}})^2 = (\sqrt[3]{27})^2 = 3^2 = 9$$

Reciprocal rule

Definition

The reciprocal of x is defined to be the number $\frac{1}{x}$, where $x \neq 0$. We can also give the reciprocal of x in index form as x^{-1}.

Consider $\left(\frac{a}{b}\right)^{-n}$. We can express this as:

$$\left(\frac{a}{b}\right)^{-n} = \left[\left(\frac{a}{b}\right)^{-1}\right]^{n} = \left[\frac{1}{a/b}\right]^{n} = \left(\frac{b}{a}\right)^{n}$$

We simplified part of this using the definition of reciprocal. Therefore:

$$\boxed{\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}}$$

Simplifying indices in algebraic expressions

Consider $\dfrac{12x^3 \times 4x^4}{x^2}$

Simplify the numerator first (using equation 1) and then carry out the division (equation 2).

$$\begin{aligned}\frac{12x^3 \times 4x^4}{x^2} &= \frac{12 \times 4 \times x^{3+4}}{x^2} \\ &= \frac{48x^7}{x^2} \\ &= 48x^{7-2} \\ &= 48x^5\end{aligned}$$

Simplifying indices in algebraic expressions

Expand and simplify the expression $\left(xy^2 - \dfrac{4}{x^2}\right)^2$

We'll expand the brackets like this:

$$\begin{aligned}
\left(xy^2 - \frac{4}{x^2}\right)\left(xy^2 - \frac{4}{x^2}\right) &= (xy^2)^2 + \left(-\frac{4}{x^2}\right)\left(-\frac{4}{x^2}\right) \\
&\quad - \frac{4}{x^2}(xy^2) - \frac{4}{x^2}(xy^2) \\
&= x^2y^4 + \frac{16}{x^4} - \frac{4xy^2}{x^2} - \frac{4xy^2}{x^2} \\
&= x^2y^4 + \frac{16}{x^4} - \frac{8y^2}{x}
\end{aligned}$$

Evaluating indices

Without a calculator — evaluate $\left(\frac{9}{100}\right)^{-\frac{1}{2}} \times \left(\sqrt{3}\right)^6$.

Simplify each term: the first term using equation (4):

$$\left(\frac{9}{100}\right)^{-\frac{1}{2}} = \left(\frac{100}{9}\right)^{\frac{1}{2}} = \frac{100^{\frac{1}{2}}}{9^{\frac{1}{2}}} = \frac{10}{3}$$

Simplify the second term using equation (3):

$$\left(\sqrt{3}\right)^6 = \left(3^{\frac{1}{2}}\right)^6 = 3^{\frac{1}{2}\times 6} = 3^3$$

Combining the results we get

$$\left(\frac{9}{100}\right)^{-\frac{1}{2}} \times \left(\sqrt{3}\right)^6 = \frac{10}{3} \times 3^3 = 10 \times 3^2 = 10 \times 9 = 90$$

Summary

Rules of indices

Below is a summary of the results shown on the previous slides.

$$\begin{aligned} a^m \times a^n &= a^{m+n} \\ a^m \div a^n &= a^{m-n} \\ (a^m)^n &= a^{mn} \end{aligned}$$

The following are also often useful:

$$a^{-n} = \frac{1}{a^n} \qquad \sqrt[n]{a} = a^{\frac{1}{n}} \qquad \sqrt[n]{a^m} = a^{\frac{m}{n}} \qquad \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

And finally, it's worth memorising these special results:

$$a^0 = 1 \qquad a^{\frac{1}{2}} = \sqrt{a} \qquad a^1 = a \qquad a^{-1} = \frac{1}{a}$$

Test yourself...

You should be able to solve the following problems if you have understood everything in these notes.

1. Express $3^5 \times 3^2$ in the form 3^n
2. Simplify the expression $\dfrac{x^{\frac{2}{3}}}{x^{\frac{1}{6}}}$
3. Express $(8^3)^6$ in the form 8^n
4. Evaluate $\left(\frac{16}{25}\right)^{\frac{1}{2}}$
5. Evaluate $\left(\frac{8}{27}\right)^{-\frac{1}{3}}$

Answers:

1. 3^7
2. $x^{\frac{1}{2}}$
3. 8^{18}
4. $\frac{4}{5}$
5. $\frac{3}{2}$