

DE MOIVRE'S THEOREM: TRIG IDENTITIES

ALGEBRA 8

INU0114/514 (MATHS 1)

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Objectives

This presentation will cover the following:

- Recap of Binomial expansions and De Moivre's theorem
- Using De Moivre's theorem to produce trig identities
 - Express multiple angle functions (e.g. $\sin 4\theta$) in terms of single angle functions $\sin \theta$ and $\cos \theta$.
 - Express powers of functions $\sin^4 \theta$ in terms of multiple angle functions (e.g. $\cos 4\theta$).

In this context De Moivre's theorem is a mathematical tool which shows us relationships between trig functions. It provides an alternative (often easier) way of simplifying trig functions that we might need to apply calculus to (e.g. integration).

Recap: De Moivre's Theorem

De Moivre's theorem is a relationship between complex numbers and trigonometry.

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

Previously we used it to evaluate powers and n^{th} roots of a number.

By expanding the LHS and comparing real and imaginary coefficients it is possible to derive trig identities for powers of sine and cosine in terms of compound angles.

We will study this application in next.

Recap: Binomial expansions

Expanding brackets

Use the binomial theorem to expand $\left(x + \frac{1}{x}\right)^4$

Recall that the binomial coefficients $n=4$ are 1, 4, 6, 4 and 1.

$$\begin{aligned}\left(x + \frac{1}{x}\right)^4 &= 1(x^4)\left(\frac{1}{x}\right)^0 + 4(x^3)\left(\frac{1}{x}\right)^1 + 6(x^2)\left(\frac{1}{x}\right)^2 \\ &\quad + 4(x^1)\left(\frac{1}{x}\right)^3 + 1(x^0)\left(\frac{1}{x}\right)^4 \\ \therefore \left(x + \frac{1}{x}\right)^4 &= x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}\end{aligned}$$

We will need the binomial series for our work with De Moivre's theorem and trig identities.

De Moivre's theorem and trig identities

Consider De Moivre's theorem for the case $n = 2$

$$(\cos \theta + i \sin \theta)^2 = \cos 2\theta + i \sin 2\theta$$

If we expand the brackets on the LHS:

$$(\cos \theta)^2 + (i \sin \theta)^2 + 2i \sin \theta \cos \theta = \cos 2\theta + i \sin 2\theta$$

$$\cos^2 \theta - \sin^2 \theta + 2i \sin \theta \cos \theta = \cos 2\theta + i \sin 2\theta$$

If we compare the real parts of both sides:

$$\cos^2 \theta - \sin^2 \theta = \cos 2\theta \quad (1)$$

Comparing the imaginary parts gives us

$$2 \sin \theta \cos \theta = \sin 2\theta \quad (2)$$

Equations (1) and (2) are trig identities which you might recognise from past work.

De Moivre's theorem provides a way of generating trig identities.

Multiple angle to single angle

Express $\cos 3\theta$ in terms of $\sin \theta$ and $\cos \theta$.

Construct an equation from De Moivre's theorem containing the angle 3θ

$$\cos 3\theta + i \sin 3\theta = (\cos \theta + i \sin \theta)^3$$

We are going to be writing a lot of 'sines' and 'cosines': so put $c = \cos \theta$ and $s = \sin \theta$.

Expand the RHS using binomial theorem:

$$\begin{aligned} \cos 3\theta + i \sin 3\theta &= (c + is)^3 \\ &= c^3 + 3c^2(is) + 3c(is)^2 + (is)^3 \\ &= c^3 + 3c^2si + 3cs^2i^2 + s^3i^3 \end{aligned}$$

Simplify the powers of i wherever possible:

$$\begin{aligned} \cos 3\theta + i \sin 3\theta &= c^3 + 3c^2si - 3cs^2 - s^3i \\ &= c^3 - 3cs^2 + (3c^2s - s^3)i \end{aligned}$$

We grouped the real and imaginary parts on the RHS.

Equating the real part on each side then we get the relationship we need:

$$\cos 3\theta \equiv \cos^3 \theta - 3 \cos \theta \sin^2 \theta$$

We didn't need it, but the method also gives us another identity for free!

Compare the imaginary parts on both sides of the equation to get:

$$\sin 3\theta \equiv 3 \cos^2 \theta \sin \theta - \sin^3 \theta$$

Useful formulae

A complex number in polar form is written $z = \cos \theta + i \sin \theta$.

Raise to the power n and apply De Moivre's theorem:

$$\begin{aligned} z^n &= (\cos \theta + i \sin \theta)^n \\ \therefore z^n &= \cos n\theta + i \sin n\theta \end{aligned} \quad (3)$$

Substituting a power of $-n$:

$$z^{-n} = (\cos \theta + i \sin \theta)^{-n} = \frac{1}{(\cos \theta + i \sin \theta)^n} = \frac{1}{\cos n\theta + i \sin n\theta}$$

Multiply top and bottom by the complex conjugate:

$$z^{-n} = \frac{1}{\cos n\theta + i \sin n\theta} \times \frac{\cos n\theta - i \sin n\theta}{\cos n\theta - i \sin n\theta} = \frac{\cos n\theta - i \sin n\theta}{\cos^2 n\theta + \sin^2 n\theta}$$

Therefore

$$z^{-n} = \frac{1}{z^n} = \cos n\theta - i \sin n\theta \quad (4)$$

Add (3) and (4) to get:

$$z^n + \frac{1}{z^n} = 2 \cos n\theta \quad (5)$$

Subtract (4) from (3):

$$z^n - \frac{1}{z^n} = 2i \sin n\theta \quad (6)$$

Deriving trig identities

Express $\sin^4 \theta$ in terms of multiple angles of $\sin \theta$ and $\cos \theta$.

Since we have a function of $\sin \theta$ we'll use equation 6. Setting $n = 1$ we can write:

$$2i \sin \theta = z - \frac{1}{z}$$

We have to make the $\sin^4 \theta$ term so:

$$(2i \sin \theta)^4 = \left(z - \frac{1}{z} \right)^4$$

Expand and simplify the LHS. Expand the RHS with the binomial theorem:

$$\begin{aligned} (2i)^4 \sin^4 \theta &= z^4 + 4z^3 \left(-\frac{1}{z} \right) + 6z^2 \left(-\frac{1}{z} \right)^2 + 4z \left(-\frac{1}{z} \right)^3 + \left(-\frac{1}{z} \right)^4 \\ 16 \sin^4 \theta &= z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4} \end{aligned}$$

We group the powers of z like this:

$$\begin{aligned} 16 \sin^4 \theta &= z^4 + \frac{1}{z^4} - 4z^2 - \frac{4}{z^2} + 6 \\ &= \left(z^4 + \frac{1}{z^4} \right) - 4 \left(z^2 + \frac{1}{z^2} \right) + 6 \end{aligned}$$

The grouped terms inside the brackets can be expressed using the cosine function given in equation 5.

Therefore:

$$\begin{aligned} 16 \sin^4 \theta &= (2 \cos 4\theta) - 4(2 \cos 2\theta) + 6 \\ &= 2 \cos 4\theta - 8 \cos 2\theta + 6 \\ \sin^4 \theta &= \frac{2 \cos 4\theta - 8 \cos 2\theta + 6}{16} \end{aligned}$$

Simplify to get

$$\sin^4 \theta = \frac{\cos 4\theta - 4 \cos 2\theta + 3}{8}$$

Deriving trig identities

Express $\cos^3 4\theta$ in terms of multiple angle functions.

Since we have a function of $\cos \theta$ we'll use equation 5. Setting $n = 4$ we can write:

$$2 \cos 4\theta = z^4 + \frac{1}{z^4}$$

We have to make the $\cos^3 4\theta$ term so:

$$(2 \cos 4\theta)^3 = \left(z^4 + \frac{1}{z^4}\right)^3$$

Expand and simplify the LHS. Expand the RHS with the binomial theorem:

$$\begin{aligned} 2^3 \cos^3 4\theta &= (z^4)^3 + 3(z^4)^2 \left(\frac{1}{z^4}\right) + 3z^4 \left(\frac{1}{z^4}\right)^2 + \left(\frac{1}{z^4}\right)^3 \\ 8 \cos^3 4\theta &= z^{12} + 3z^4 + \frac{3}{z^4} + \frac{1}{z^{12}} \end{aligned}$$

We group the powers of z like this:

$$8 \cos^3 4\theta = z^{12} + \frac{1}{z^{12}} + 3\left(z^4 + \frac{1}{z^4}\right)$$

The grouped terms inside the brackets can be expressed using the cosine function given in equation 5.

Therefore:

$$8 \cos^3 4\theta = (2 \cos 12\theta) + 3(2 \cos 4\theta)$$

$$8 \cos^3 4\theta = 2 \cos 12\theta + 6 \cos 4\theta$$

$$4 \cos^3 4\theta = \cos 12\theta + 3 \cos 4\theta$$

Simplify to get

$$\cos^3 4\theta = \frac{\cos 12\theta + 3 \cos 4\theta}{4}$$

Summary

To express multiple angle functions in terms of $\sin \theta$ and $\cos \theta$ use de Moivre's theorem

$$\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n$$

and expand the RHS (using binomial series if necessary) and compare the real or imaginary parts to obtain an identity.

To express $\sin^k \theta$ or $\cos^k \theta$ in terms of multiple angle functions use the formulae choose either

$$2 \cos \theta = z + \frac{1}{z} \quad \text{or} \quad 2i \sin \theta = z - \frac{1}{z}$$

and then use binomial expansions (if necessary) to find $\sin^k \theta$ or $\cos^k \theta$ in terms of z . Then use these formulas

$$z^n + \frac{1}{z^n} = 2 \cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i \sin n\theta$$