De Moivre's Theorem: powers and roots

Algebra 8

INU0114/514 (Maths 1)

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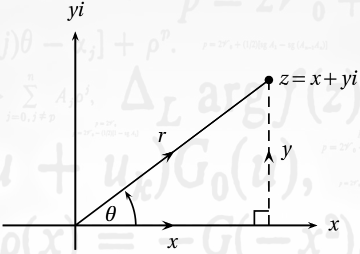
Objectives

This presentation will cover the following:

- Recap of polar and exponential form
- Understand more clearly the argument of a complex number
- De Moivre's theorem
- Using De Moivre's theorem to find powers and roots

Recap: polar and Cartesian form

Complex numbers can be represented in polar form on the Argand diagram.



We can convert from polar to Cartesian form using

$$x = r\cos\theta \quad \text{and } y = r\sin\theta$$

And from Cartesian form to polar form using

$$r = \sqrt{x^2 + y^2} \quad \text{and } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

where the *argument* θ must be in the correct quadrant.

A nice pattern...

Consider a complex number z in polar form:

$$z = \cos\theta + i\sin\theta$$

Let's calculate z^2 by squaring both sides:

$$z^2 = (\cos\theta + i\sin\theta)^2$$

Expand the brackets on the RHS:

$$\begin{aligned} z^2 &= \cos^2\theta + i^2\sin^2\theta + 2i\sin\theta\cos\theta \\ &= \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta \end{aligned}$$

We know from trigonometry that $\cos^2\theta - \sin^2\theta \equiv \cos 2\theta$ and $2\sin\theta\cos\theta \equiv \sin 2\theta$.

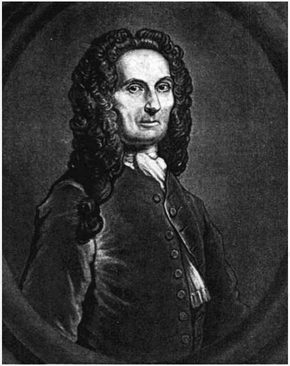
Therefore: $z^2 = \cos 2\theta + i\sin 2\theta$

To summarise:

$$(\cos\theta + i\sin\theta)^2 \equiv \cos 2\theta + i\sin 2\theta$$

We can take the power and multiply it with the angle. But is this true for any power/exponent?

De Moivre's Theorem



Abraham de Moivre (1667 — 1754)

De Moivre's theorem is a relationship between complex numbers and trigonometry.

Here it is:

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

By expanding the LHS and comparing real and imaginary coefficients it is possible to derive trig identities for powers of sine and cosine in terms of compound angles — very useful in calculus.

De Moivre's theorem also makes it possible to quickly evaluate powers and n^{th} roots of complex numbers.

Evaluating powers

Calculating powers

Given the complex number

$$z = 1 + \sqrt{3}i$$

Use De Moivre's theorem to evaluate z^5.

The modulus is $|z| = 2$ and $\arg z = \frac{\pi}{3}$.

The polar form of z is

$$z = 2(\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3})$$

Therefore:

$$\begin{aligned} z^5 &= \left[2(\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3})\right]^5 \\ &= 32(\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3})^5 \end{aligned}$$

By De Moivre's theorem:

$$z^5 = 32(\cos\tfrac{5\pi}{3} + i\sin\tfrac{5\pi}{3})$$

Simplify to get the Cartesian form:

$$\begin{aligned} z^5 &= 32(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i) \\ &= 16 - 16\sqrt{3}i \end{aligned}$$

De Moivre's theorem and n^{th} roots

De Moivre's theorem:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

can also be expressed in this form:

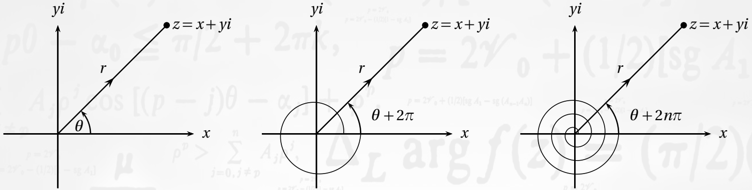
$$(\cos\theta + i\sin\theta)^{\frac{1}{n}} = \cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)$$

In this case – n is still a positive integer, but the exponent now represents a root. For example, $n = 2$ represents a square-root.

The fundamental theorem of algebra tells to expect n roots — not just one.

How do we find the others? We use the periodic nature of sine and cosine...

The argument revisited



The argument of a complex number $\arg z = \theta$ is the angle made by the line to positive x-axis (first picture).

The angle θ can be increased by 2π and it will still describe the same line in the xy-plane (second picture).

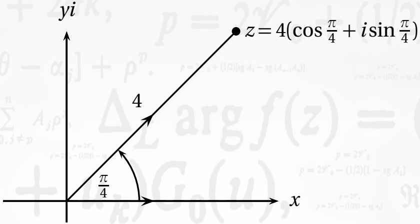
Any integer multiple of 2π can be added (or subtracted) to θ without changing the Cartesian form of the complex number.

These statements are simply a consequence of the periodic nature of sine and cosine:

$$\cos(\theta + 2n\pi) \equiv \cos\theta \quad \text{and} \quad \sin(\theta + 2n\pi) \equiv \sin\theta$$

Consider the complex number $z = 4(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = 4\angle\frac{\pi}{4}$.

Here is the complex number on an Argand diagram



If we add 2π (or 4π or 6π or...$2n\pi$) to the angle the point on the Argand diagram will remain unchanged.

In this case z is also equivalent to

$$z = 4\angle\frac{9\pi}{4} \quad , z = 4\angle\frac{17\pi}{4} \quad , z = 4\angle\frac{25\pi}{4}$$

The angle $\frac{\pi}{4}$ is the principal value.

But adding multiples of 2π will give an equivalent angle.

The general polar form of a complex number is

$$z = r(\cos(\theta + 2\pi n) + i\sin(\theta + 2\pi n)) = r\angle(\theta + 2\pi n)$$

and the general exponential form is

$$z = e^{i(\theta + 2\pi n)}$$

We have been restricting the allowable values to $0 \le \theta < 2\pi$. But as we saw earlier, other values of θ are also valid.

Multivalued functions?

Previously we saw how to calculate logarithms of complex numbers. For example

$$\ln(-10) = \ln 10 + \pi i$$

Recall that the definition of a function states that one input (in the domain) should give one output value (in the range). We usually don't allow other values — like $\ln 10 + 3\pi i$ — in order to satisfy that definition.

We do however, need to consider those other values of θ to obtain n^{th} roots using De Moivre's theorem.

Calculation of n^{th} roots

Definition

A number w is a n^{th} root of a number z if $w^n = z$.

A few statements to clarify this definition...

- The number 2 is the 4^{th} root of 16 because $2^4 = 16$.
- The number –2 is also the 4^{th} root of 16 because $(-2)^4 = 16$.
- The number 3 is the cube root of 27 because $3^3 = 27$.

From the previous example:

- The number $1+\sqrt{3}i$ is the 5^{th} root of $16-16\sqrt{3}i$

The fundamental theorem of algebra tells to expect n roots — not just one. De Moivre's theorem in this case is

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta+2\pi k}{n}\right)+i\sin\left(\frac{\theta+2\pi k}{n}\right)\right)$$

where k is an integer (and $k = 0, 1, 2 \ldots n-1$).

Cube roots of a complex number

Find all of the cube roots of $27i$.

This means we are looking for complex numbers z with the property

$$z = 27i$$

First, we write z in polar form

$$z = 27(\cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2})$$

Take cube roots and apply De Moivre's theorem:

$$\begin{aligned} z^{\frac{1}{3}} &= \left[27(\cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2})\right]^{\frac{1}{3}} \\ &= 3(\cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2})^{\frac{1}{3}} \\ z^{\frac{1}{3}} &= 3(\cos\tfrac{\pi}{6} + i\sin\tfrac{\pi}{6}) \end{aligned}$$

In Cartesian form this is $z^{\frac{1}{3}} = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$.

However, there are other values of z which give $z = 27i$.

We wrote $27i$ in polar form as

$$27i = 27(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$$

but we could also add 2π to the angle and it would still be $27i$:

$$27i = 27(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2})$$

Therefore we have

$$27i = z = 27(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2})$$

If we apply De Moivre's theorem to this:

$$\begin{aligned} z^{\frac{1}{3}} &= \left[27(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2})\right]^{\frac{1}{3}} \\ z^{\frac{1}{3}} &= 3(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}) \end{aligned}$$

In Cartesian form this is $z^{\frac{1}{3}} = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i$.

We found another cube root — different to the first!

Add another 2π to the polar form for $27i$ and we get

$$\begin{aligned} z &= 27(\cos\tfrac{9\pi}{2} + i\sin\tfrac{9\pi}{2}) \\ z^{\frac{1}{3}} &= 27(\cos\tfrac{9\pi}{2} + i\sin\tfrac{9\pi}{2})^{\frac{1}{3}} \\ \therefore z^{\frac{1}{3}} &= 3(\cos\tfrac{3\pi}{2} + i\sin\tfrac{3\pi}{2}) \end{aligned}$$

In Cartesian form this is $z^{\frac{1}{3}} = -3i$.

We've found three cube roots now. Are there any more?

Adding another 2π gives us

$$\begin{aligned} z &= 27(\cos\tfrac{13\pi}{2} + i\sin\tfrac{13\pi}{2}) \\ \therefore z^{\frac{1}{3}} &= 3(\cos\tfrac{13\pi}{6} + i\sin\tfrac{13\pi}{6}) \\ z^{\frac{1}{3}} &= \tfrac{3\sqrt{3}}{2} + \tfrac{3}{2}i \end{aligned}$$

This is a repeat of the first root. If we add more multiples of 2π we will also duplicate the other roots.

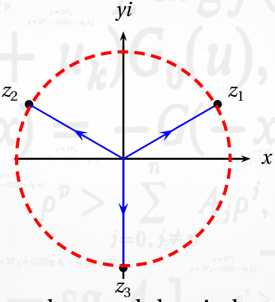
The number $27i$ has three cube roots. They are

$$z_1 = \frac{3\sqrt{3}}{2} + \frac{3}{2}i \quad z_2 = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i \quad z_3 = -3i$$

Or in polar form:

$$z_1 = 3\angle\frac{\pi}{6}, \quad z_2 = 3\angle\frac{5\pi}{6}, \quad z_3 = 3\angle\frac{3\pi}{2}$$

On an Argand diagram we can see an obvious pattern with these roots:



The roots are equally spaced around the circle.

The angle between the roots on the diagram is $\frac{2\pi}{3}$. The radius of the circle is $|z| = 3$.

De Moivre's Theorem and n^{th} roots.

Calculate all of the 4^{th} roots of -64

We are trying to find $z^{\frac{1}{4}}$ where $z=-64$. There will be four roots.

In polar form we write $-64 = 64(\cos\pi + i\sin\pi) = 64\angle\pi$.

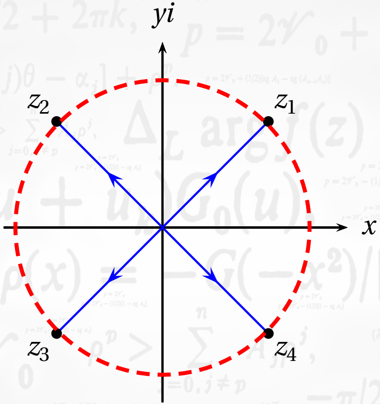
Therefore the four equations we must solve are:

$$\begin{aligned} z &= 64\angle\pi \\ z &= 64\angle 3\pi \\ z &= 64\angle 5\pi \\ z &= 64\angle 7\pi \end{aligned}$$

Applying De Moivre's theorem to these gives the four roots:

$$\begin{aligned} z^{\frac{1}{4}} &= [64\angle\pi]^{\frac{1}{4}} = \sqrt{8}\angle\tfrac{\pi}{4} = 2+2i \\ z^{\frac{1}{4}} &= [64\angle 3\pi]^{\frac{1}{4}} = \sqrt{8}\angle\tfrac{3\pi}{4} = -2+2i \\ z^{\frac{1}{4}} &= [64\angle 5\pi]^{\frac{1}{4}} = \sqrt{8}\angle\tfrac{5\pi}{4} = -2-2i \\ z^{\frac{1}{4}} &= [64\angle 7\pi]^{\frac{1}{4}} = \sqrt{8}\angle\tfrac{7\pi}{4} = 2-2i \end{aligned}$$

The Argand diagram of the 4th roots of −64 looks like this:



The roots are equally spaced around the circle; the angle between them is $\frac{2\pi}{4} = \frac{\pi}{2}$. The radius of the circle is $|z| = \sqrt{8}$.

Summary

Definition

A number w is a n^{th} root of a number z if $w^n = z$.

In general, to find the n^{th} root of a complex number:

- There will be n roots
- The n roots will be equally spaced by an angle $\frac{2\pi}{n}$.
- The polar form will be

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right)$$

where r is the modulus and θ is the principal value (of the first root).