# Polar and exponential forms

## Algebra 7

INU0114/514 (Maths 1)

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# Objectives

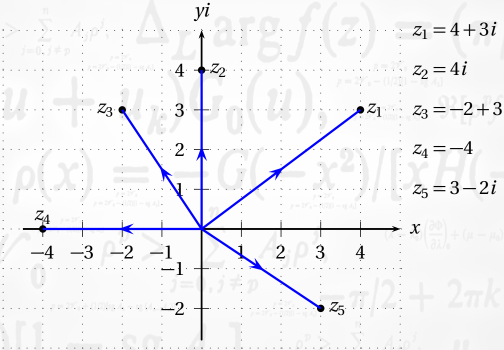
This presentation will cover the following:

- Argand diagrams and the complex plane
- Modulus and argument
- Conversion between Cartesian, polar and exponential form
- Logarithm of a complex number

This will prepare us for later topics such as De Moivre's theorem.

# Argand diagram

We've previously seen that complex numbers of the form $x + yi$ are said to be in *Cartesian form* because they can be represented as points $(x, y)$ on a graph called an Argand diagram.




This means we can examine complex numbers using the rules of geometry, trigonometry and algebra.

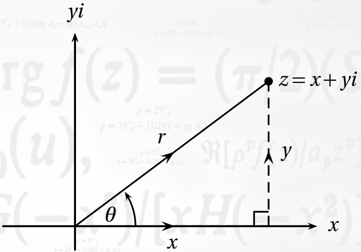
# Polar form of a complex number

Argand diagrams show us how to express complex numbers in *polar form.*

A complex number in polar form is defined by its modulus (length) and argument (angle).

The modulus is the distance $r$ of the point from the origin.

The argument is an angle $\theta$ measured anticlockwise from the positive $x$-axis.




In polar form the complex number $z = x + yi$ becomes

$$z = r\cos\theta + (r\sin\theta)i$$

The *modulus* $r$ and *argument* $\theta$ is given by

$$r = \sqrt{x^2 + y^2} \quad \text{and } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

and $\theta$ must be in the correct quadrant.

## Cartesian to polar form

Express $z = -4 + i$ in polar form. Give the argument, in radians to 3 S.F.

In this case we have $|z| = \sqrt{17}$ .

Since $z$ is in the second quadrant then $\arg z = \pi + \tan^{-1}(-\frac{1}{4}) = 2.897$ (radians).

Therefore

$$z = \sqrt{17}(\cos 2.897 + i \sin 2.897)$$

A quick way of representing the polar form is to use the $r \operatorname{cis} \theta$ or $r\angle\theta$ notations.

For example $z = \sqrt{17} \operatorname{cis} 2.897$ or $z = \sqrt{17}\angle 2.897$.

## Polar to cartesian form

Express $z = 10\angle\frac{7\pi}{6}$ in cartesian form.

In this case we have $|z| = 10$ and $\theta = \frac{7\pi}{6}$.

Therefore

$$\begin{aligned} z &= 10(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}) \\ &= 10(-\frac{\sqrt{3}}{2} + i(-\frac{1}{2})) \\ \therefore z &= -5\sqrt{3} - 5i \end{aligned}$$

# Exponential form

Consider the Maclaurin series for the exponential function:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \dots$$

Substitute $x = i\theta$ to get a series for $e^{i\theta}$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} \dots$$

Expanding the brackets and recalling that $i^2 = -1$, $i^3 = -i$, etc:

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} \dots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} \dots \end{aligned}$$

Grouping the real and imaginary terms we get:

$$\mathrm{e}^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\ldots\right)$$

The terms within the first set of brackets are the terms for the infinite series for $\cos\theta$.

The terms within the second set of brackets are the terms of the infinite series for the function $\sin\theta$.

Therefore:

$$\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$$

This relationship is known as Euler's Identity.

If we multiply both sides of this by the modulus $r$ we obtain a slightly more general form:

$$\boxed{r\mathrm{e}^{i\theta} = r(\cos\theta + i\sin\theta)}$$

## Cartesian to polar and exponential form

Express $z = \sqrt{3} - i$ in polar and exponential form.

The modulus is

$$|z| = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$$

The argument is in the fourth quadrant so that

$$\arg z = \tan(\frac{-1}{\sqrt{3}}) + 2\pi = \frac{11\pi}{6}$$

The polar form of the number is

$$z = 2(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6})$$

and the exponential form is

$$z = 2e^{i\frac{11\pi}{6}}$$

# Powers of complex numbers

Changing a complex number to exponential or polar form makes it easy to evaluate powers of that number.

## Evaluating a power

Evaluate $(1-i)^4$.

We can write $1-i$ in exponential form as

$$1-i=\sqrt{2}e^{\frac{7\pi}{4}i}$$

Raising both sides to the power of 4:

$$(1-i)^4 = (\sqrt{2}e^{\frac{7\pi}{4}i})^4 = 4e^{7\pi i}$$

Changing back to cartesian form (via polar form):

$$\begin{aligned}(1-i)^4 &= 4(\cos 7\pi + i\sin 7\pi)\\ &= 4(-1)\\ \therefore (1-i)^4 &= -4\end{aligned}$$

For higher powers (or $n^{\text{th}}$ roots) it may be necessary to use De Moivre's theorem. We'll study this method at a later time.

# Logarithms of complex numbers

The function $\ln x$ for $x \in \mathbb{R}$ is defined on the domain $x > 0$, but we can extend the domain to $x \in \mathbb{C}$.

**Complex logarithm**

Evaluate $\ln(1+\sqrt{3}i)$.

Express $1+\sqrt{3}i$ in exponential form. Here we have $r = 2$ and $\theta = \frac{1}{3}\pi$.

Therefore

$$\ln(1+\sqrt{3}) = \ln\left(2e^{i\frac{\pi}{3}}\right)$$

Use log rules to simplify:

$$\begin{aligned} &= \ln 2 + \ln e^{i\frac{\pi}{3}} \\ \ln(1+\sqrt{3}) &= \ln 2 + i\frac{\pi}{3} \end{aligned}$$

# Multivalued functions

In the previous example the angle calculated was the principal value: $\frac{\pi}{3}$.

We could also have chosen $\theta = \frac{1}{3}\pi + 2\pi = \frac{7}{3}\pi$ to represent the same logarithm value. In general, the complex logarithm can take the values specified by

$$\ln(x+yi) = \ln\left(\sqrt{x^2+y^2}\right) + \arg(x+yi) + 2n\pi, \qquad n \in \mathbb{Z}$$

Although the exponential (and polar) forms differ by multiples of $2\pi$ — the Cartesian forms are all identical.

The complex logarithm is sometimes called "a multivalued function". As we saw in Semester 1, functions are supposed to have one input and one output so this name is a misnomer!

We will return to this concept with De Moivre's theorem in the final part of the course.

Other examples of multivalued functions are square-roots, in which every real number is associated with two roots.

# Finally...a beautiful equation!

Given the relationship

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$

If we put $r = 1$ and $\theta = \pi$ into we obtain

$$(1)e^{i\pi} = e^{i\pi}(1)(\cos\pi + i\sin\pi) \qquad \Rightarrow e^{i\pi} = -1$$

Rearranging we can make this equation

$$\boxed{e^{i\pi} + 1 = 0}$$

This is called Euler's Formula and it is often cited as being one of the most beautiful discoveries in all of mathematics.

It relates, in a single equation, five of the most important numbers in mathematics.

# Test yourself...

Given the complex numbers

$$z_1 = 5i \quad z_2 = -5 - 12i \quad z_3 = 10e^{i\frac{3\pi}{4}}$$

1. Find $|z_1|$ and $\arg z_1$
2. Express $z_2$ in polar and exponential form.
3. Express $z_3$ in Cartesian form.
4. Calculate $z_2 z_3$ in polar form
5. Calculate $\ln z_3$
6. Calculate $\ln(-2)$

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Answers:

1. $|z_1| = 5$ and $\arg z_1 = \frac{\pi}{2}$
2. $z_2 = 13(\cos 4.318 + i \sin 4.318)$ and $z = 13e^{4.318i}$
3. $z_3 = -5\sqrt{2} + 5\sqrt{2}i$
4. $z_2 z_3 = (13)(10)e^{i(4.318 + \frac{3\pi}{4})} = 130e^{6.674i}$
5. $\ln z_3 = \ln 10 + \frac{3}{4}\pi i$
6. $-2 = 2e^{i\pi}$. Therefore $\ln(-2) = \ln 2 + \pi i$