Complex Numbers

Algebra 7

INU0114/514 (Maths 1)

Dr Adrian Jannetta MIMA CMath FRAS

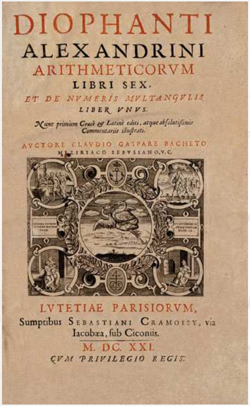
INTO Newcastle University

Objectives

This presentation will cover the following:

- Introduction to complex numbers.
- Argand diagrams
- Arithmetic of complex numbers.
- Complex roots of a quadratic equation
- Conjugate of complex number
- Equal complex numbers
- Algebraic method to find square-roots of a complex number

Background: solving equations



Arithmetica: an Ancient Greek text on mathematics written by the mathematician Diophantus in the 3rd century AD.

Historically mathematicians have sometimes needed new types of numbers to solve equations.

The simplest equations such as

$$2x + 5 = 25$$

have solutions in the positive integers (here, $x = 10$).

The equation

$$3x + 6 = 0$$

requires another type of number: the negative integers (here, $x = -2$).

Indian mathematician Brahmagupta was the first to give rules to compute with zero in the 7th century AD

The equation

$$x+6=6$$

requires a value between positive and negative integers: $x=0$.

The equation

$$10x=5$$

have no integers solutions: we need rational numbers. In this case: $x=\frac{1}{2}$.

Some equations don't have solutions with any the numbers found so far:

$$x^2=2$$

We need the irrational numbers (here $x=\sqrt{2}$).

For a long time, it was believed there were no more types of number to be found.

Background: solving the cubic

In the 16th century several mathematicians were trying find to a method to solve cubic equations like this:

$$x^3 - 15x - 4 = 0$$

It has one solution: $x = 4$ (check it!)

However, the cubic formula invented by mathematicians led to a strange result:

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

$\sqrt{-121}$ was taken to be a sign the formula was wrong. But Rafael Bombelli (shown here) used algebra to show that a solution could still be found. E.g. by simplifying it is possible to show

$$x = 2 + \sqrt{-1} + 2 - \sqrt{-1}$$

where the strange roots cancel to give $x = 4$; the solution.

During the next two hundred years many mathematicians explored the properties of $\sqrt{-1}$.

Imaginary numbers

There is a number that when multiplied with itself gives -1.

The basis of these numbers is denoted by i and defined to be

$$i^2 = -1$$

so that $i = \sqrt{-1}$.

Historically, these numbers came to be called imaginary numbers (and by implication the more familiar numbers are called real numbers).

We can use normal algebra rules to simplify complex numbers.

$$\sqrt{-49} = \sqrt{49}\sqrt{-1} = 7\sqrt{-1} = 7i$$

Or simplify powers of i like this:

$$i^5 = i^2 \times i^2 \times i = (-1)(-1)i = i$$

The imaginary number can be manipulated in much the same way that other algebraic symbols can be.

Recap: roots of a quadratic equation

In previous work with quadratic equations

$$ax^2 + bx + c = 0$$

The discriminant $\Delta = b^2 - 4ac$ was used to classify the roots of quadratic.

- $\Delta > 0$ meant two real and different roots.
- $\Delta = 0$ meant two real and equal roots.
- $\Delta < 0$ meant complex roots.

We are now in a position to be able to evaluate the complex roots. We'll use the quadratic formula to do this.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Let's see an example.

Complex roots of a quadratic

Solve the equation

$$x^2 + 8x + 25 = 0$$

Using the quadratic formula:

$$x = \frac{-8 \pm \sqrt{8^2 - 4(1)(25)}}{2(1)} = \frac{-8 \pm \sqrt{-36}}{2}$$

But $\sqrt{-36} = \sqrt{36}\sqrt{-1} = 6i$ so the solution can be further simplified:

$$x = \frac{-8 \pm 6i}{2} = -4 \pm 3i$$

The two solutions are $x = -4 - 3i$ and $x = -4 + 3i$.

Quadratic equation

Solve the equation

$$3x^2 + 2x + 2 = 0$$

Using the quadratic formula:

$$x = \frac{-2 \pm \sqrt{2^2 - 4(3)(2)}}{2(3)} = \frac{-2 \pm \sqrt{-20}}{6}$$

But $\sqrt{-20} = \sqrt{4}\sqrt{-5} = 2\sqrt{5}i$

$$x = \frac{-2 \pm 2\sqrt{5}i}{6} = -\frac{1}{3} \pm \frac{\sqrt{5}}{3}i$$

The two solutions are $x = -\frac{1}{3} - \frac{\sqrt{5}}{3}i$ and $x = -\frac{1}{3} + \frac{\sqrt{5}}{3}i$.

Complex Numbers

The solutions of $x^2 + 8x + 25 = 0$ are $x = -4 \pm 3i$.

The solutions are a mixture of real and imaginary numbers and are called complex numbers.

Complex number

Complex numbers, often denoted by the letter z, have the form

$$z = x + yi$$

where x and y are real numbers and $i = \sqrt{-1}$.

Complex numbers have a real part and imaginary part.

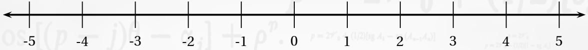
When $x = 0$: z is an imaginary number.

When $y = 0$: z is a real number.

That means all real numbers are also complex numbers. The set of complex numbers is denoted by $\mathbb{C}$ (as opposed to $\mathbb{R}$ for the set of real numbers).

Argand diagrams

Here is number line; a picture in which every point corresponds to a real number.

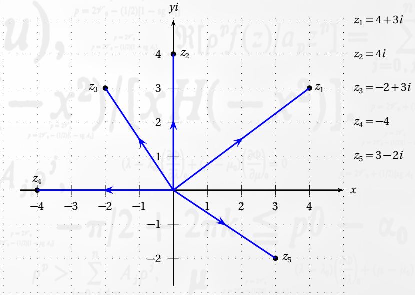


Complex numbers can be represented on a graph called an *Argand diagram*

Every point (x, y) on the graph can represent a complex number $z = x + yi$.

The x-axis represents the real number-line; it is called the *real axis*. The yi-axis is called the *imaginary axis*.

Complex numbers can be located above, below or on the real axis.



Argand diagrams suggest complex numbers can analysed with geometry and trigonometry (as we did with vectors) as well as with algebra.

Arithmetic with complex numbers

Complex numbers obey the usual rules of algebra.

Addition and subtraction acts on the corresponding real and imaginary parts. Consider the complex numbers z_1 and z_2 where

$$z_1 = 3 + 5i \quad \text{and} \quad z_2 = -4 + 10i$$

Addition: $z_1 + z_2 = (3 - 4) + i(5 + 10) = -1 + 15i$

Subtraction: $z_2 - z_1 = (-4 - 3) + i(10 - 5) = -7 + 5i$

Multiplication by a scalar:

$$2z_1 = 2(3 + 5i) = (2)(3) + (2)(5i) = 6 + 10i$$

Division by a scalar is also easy; it acts on the components separately:

$$\frac{z_2}{2} = \frac{-4 + 10i}{2} = \frac{-4}{2} + \frac{10i}{2} = -2 + 5i$$

Multiplication of two complex numbers involves multiplying all components together and adding.

We also remember that, by definition, $i^2 = -1$.

$$\begin{aligned} z_1 z_2 &= (3+5i)(-4+10i) \\ &= 3(-4+10i)+5i(-4+10i) \\ &= -12+30i-20i+50i^2 \\ &= -12+10i+50(-1) \\ &= -12+10i-50 \\ \therefore z_1 z_2 &= -62+10i \end{aligned}$$

To calculate powers, for example z^5, we could expand using binomial series. However, we'll see a more effective method in the next presentation.

Equal complex numbers

Two complex numbers

$$z_1 = x + yi \quad \text{and } z_2 = a + bi$$

are equal if the real parts and imaginary parts are both separately equal.

In other words, if

$$x = a \text{ and } y = b$$

then $z_1 = z_2$.

This is a simple idea but very powerful when manipulating complex numbers.

Equating real and imaginary parts

Find the real values of x and y in the equation

$$\frac{x}{2-i} + \frac{yi}{3+i} = 4$$

Put the terms on the LHS over a common denominator

$$\frac{x(3+i)+yi(2-i)}{(2-i)(3+i)} = 4$$

Cross multiply to get

$$x(3+i)+yi(2-i) = 4(2-i)(3+i)$$

Expand the brackets and group the real and imaginary parts on each side:

$$\begin{aligned} 3x+ix+2yi-yi^2 &= 4(7-i) \\ 3x+y+(x+2y)i &= 28-4i \end{aligned}$$

$$3x + y + (x + 2y)i = 28 - 4i$$

We have equal complex numbers here.

The real part of the LHS must be equal to the real part on the RHS.

That means:

$$3x + y = 28$$

The imaginary parts on each side are also equal:

$$x + 2y = -4$$

We have two equations with two variables; solve them simultaneously.

The solutions are

$$x = 12 \text{ and } y = -8$$

Conjugate pairs

Complex conjugate

If $x+yi$ is denoted by z then its *complex conjugate*, $x-yi$ is denoted by $z^{\star}$.

Multiplying conjugate pairs always produces a real number.

For example, $4+3i$ and $4-3i$ are a conjugate complex numbers.

$$\begin{aligned}(4+3i)(4-3i) &= 16+12i-12i-9i^2\\ &= 16-9(-1)\\ &= 25\end{aligned}$$

Conjugates are useful because they allow us to carry out division of complex numbers.

Division of complex numbers

Division of two complex numbers can be done using the conjugate definition.

Division

Simplify $\dfrac{6+8i}{1-i}$ to the form $x+yi$.

Multiply top and bottom of the fraction by the conjugate of the denominator:

$$\begin{aligned}
\frac{6+8i}{1-i} &= \frac{6+8i}{1-i}\times\frac{1+i}{1+i} \\
&= \frac{6+8i+6i+8i^2}{1-i^2} \\
&= \frac{6+14i+8(-1)}{1-(-1)} = \frac{-2+14i}{2} \\
\therefore \frac{6+8i}{1-i} &= -1+7i
\end{aligned}$$

This similar to how we rationalised surds much earlier in the course.

Square-roots of complex numbers

Square-root of a complex number

Evaluate $\sqrt{16+30i}$

The result of this will be another complex number $x+yi$, so we let

$$\begin{aligned} x+yi &= \sqrt{16+30i} \\ \therefore (x+yi)^2 &= 16+30i \end{aligned}$$

We must find the values of x and y.

Expand the brackets:

$$x^2-y^2+2xyi=16+30i$$

Equate the real and imaginary parts on each side:

$$\begin{aligned} x^2-y^2 &= 16 \\ 2xy &= 30 \end{aligned}$$

We have to solve these simultaneous equations.

$$\begin{aligned} x^2 - y^2 &= 16 \\ 2xy &= 30 \end{aligned}$$

Rearrange the second equation:

$$y = \frac{15}{x}$$

Substitute into the first equation:

$$\begin{aligned} x^2 - \left(\frac{15}{x}\right)^2 &= 16 \\ x^2 - \frac{225}{x^2} &= 16 \end{aligned}$$

Multiply through by x^2 and rearrange:

$$x^4 - 16x^2 - 225 = 0$$

This quartic equation is solved by putting $u = x^2$:

$$\begin{aligned} u^2 - 16u - 225 &= 0 \\ (u-25)(u+9) &= 0 \end{aligned}$$

Therefore

$$u = 25 \text{ and } u = -9$$

which means

$$x^2 = 25 \text{ and } x^2 = -9$$

The second equation doesn't have real solutions because x is a real number.

The first equation gives $x = \pm 5$.

When $x = 5$, $y = 3$.

When $x = -5$, $y = -3$.

The values of $\sqrt{16+30i}$ are

$$-5-3i \text{ and } 5+3i$$

Theorem (Fundamental theorem of Algebra)

Every single variable polynomial of degree n has at least one complex root, and n roots in total.

Carl Gauss (1777 — 1855)

This result, discovered by Carl Gauss, means that:

- Linear equations always have one solution.
- Quadratics always have two solutions.
- Cubics always have three solutions
- Quartic equations have four solutions...etc.

All polynomial equations have real and/or complex roots.

Test yourself...

You should be able to solve the following problems based on the material covered so far.

1. Simplify i^6 and $\frac{1}{i^3}$
2. Simplify $(4+2i)(4-2i)$
3. Solve $x^2-2x+5=0$.
4. Given $z=-3-8i$, find $z^\star$
5. Simplify $(1+i)^3$
6. Simplify $(3-2i)^{-1}$
7. Find $\sqrt{-3-4i}$

Answers:

1. $i^6=-1$ and $\frac{1}{i^3}=i$
2. 20
3. $x=1\pm 2i$
4. $z^\star=-3+8i$
5. $-2+2i$
6. Use division: $\frac{1}{3-2i}=\frac{3}{13}+\frac{2}{13}i$
7. $\pm(1-2i)$