LINEAR SYSTEMS OF EQUATIONS

ALGEBRA 6

INU0114/514 (MATHS 1)

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Introduction

Matrices have numerous practical applications in mathematics and physics. One particular application involves the solution of linear systems of equations such as

$$\begin{aligned} 2x+y &= 8 \\ 4x-5y &= -4 \end{aligned}$$

These can be recast in matrix form like this:

$$\begin{pmatrix} 2 & 1 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 8 \\ -4 \end{pmatrix}$$

Or more compactly as

$$\mathbf{A}\mathbf{x}=\mathbf{b}$$

where **A** is matrix of coefficients and **b** represents the right-hand side of the equations.

In this presentation we will examine a method for solving the equations using matrix methods.

Determinants

Every square matrix has a number called a determinant associated with it.

The determinant is calculated using the elements of the matrix.

The 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has a determinant which is calculated as follows:

$$\det \mathbf{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The determinant is sometimes denoted by $|\mathbf{A}|$.

Find the determinant of the matrix $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}$.

In this case the determinant is

$$\det \mathbf{B} = \begin{vmatrix} 3 & 1 \\ -2 & 4 \end{vmatrix} = (3 \times 4) - (1 \times -2) = 12 + 2 = 14$$

Find the determinant of the matrix $\mathbf{P} = \begin{pmatrix} -10 & 4 \\ -5 & 2 \end{pmatrix}$.

In this case the determinant is

$$\det \mathbf{P} = \begin{vmatrix} -10 & 4 \\ -5 & 2 \end{vmatrix} = (-10 \times 2) - (4 \times -5) = -20 - (-20) = 0$$

Singular and nonsingular matrices

The determinant provides important information about the matrix. A matrix whose determinant is zero is said to be singular. A matrix whose determinant is not zero is said to be nonsingular.

For example, the equations

$$\begin{aligned} 2x+3y &= 10 \\ 4x-5y &= 16 \end{aligned}$$

can be represented in matrix form by

$$\begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 10 \\ 16 \end{pmatrix}$$

The determinant of the matrix of coefficients tells us whether or not there are solutions to original set of equations.

Here, we have

$$\mathbf{A}=\begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix} \quad \therefore \det\mathbf{A}=-22$$

Since **A** is nonsingular then the two equations do have solutions. If **A** had been singular ($\det\mathbf{A}=0$) then it would not have solutions.

Solutions of linear systems

Determine whether or not the equations

$$\begin{aligned} -2.5x + 13y &= 7.25 \\ 35x - 182y &= -101.5 \end{aligned}$$

have a unique solution.

In matrix form the equations are $\mathbf{Ax} = \mathbf{b}$:

$$\begin{bmatrix} -2.5 & 13 \\ 35 & -182 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7.25 \\ -101.5 \end{bmatrix}$$

The matrix of coefficients will tell us about the solution — the RHS of the equation is unimportant.

$$\mathbf{A} = \begin{bmatrix} -2.5 & 13 \\ 35 & -182 \end{bmatrix}$$

The determinant is

$$\det \mathbf{A} = (-2.5)(-182) - (13)(35) = 455 - 455 = 0$$

Therefore, there are no unique solutions to the original equations.

Inverse matrix (2×2 case)

The matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has an inverse matrix $\mathbf{A}^{-1}$ given by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

An easy way to remember the formula:

1. Swap the elements on the leading diagonal.
2. Reverse the signs of the other two elements.

The matrix $\mathbf{A}$ will ***not*** have an inverse matrix if $\det \mathbf{A} = 0$ (i.e. if the matrix is singular).

2×2 inverse matrix.

Given $\mathbf{A} = \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$ find $\mathbf{A}^{-1}$.

We need the determinant:

$$\det \mathbf{A} = (3)(5) - (7)(2) = 1$$

Using the formula given previously:

$$\mathbf{A}^{-1} = \frac{1}{1} \begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix}$$

A simple check can be carried out to see if this is correct: check that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

2×2 inverse matrix.

Given $\mathbf{P} = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}$ find $\mathbf{P}^{-1}$.

We need the determinant:

$$\det \mathbf{P} = (6)(3) - (-1)(2) = 20$$

Using the formula given previously:

$$\mathbf{P}^{-1} = \frac{1}{20} \begin{pmatrix} 3 & 1 \\ -2 & 6 \end{pmatrix}$$

(Leave the factor of $\frac{1}{20}$ outside the matrix. Much neater that way!)

Solving linear systems

Linear systems of equations in matrix form look like this:

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

Remember – this matrix equation represents many individual equations.

To make **x** the subject we cannot use division – it doesn't exist for matrices. Instead: pre-multiply both sides by the inverse of **A**:

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Remember that a matrix multiplied by its inverse gives the identity matrix:

$$\mathbf{I}\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

And the identity multiplied by a matrix leaves it unchanged:

$$\boxed{\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}}$$

Solving a system of equations

Given the equations

$$\begin{aligned} x-3y &= 6 \\ 2x+y &= 5 \end{aligned}$$

Use matrices to find the values of x and y.

The equations in matrix form

$$\begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

Or $\mathbf{Ax}=\mathbf{b}$, where

$$\mathbf{A}=\begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}, \mathbf{x}=\begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \mathbf{b}=\begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

The solutions to the system are found from $\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}$:

$$\mathbf{x}=\frac{1}{7}\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 5 \end{pmatrix}=\frac{1}{7}\begin{pmatrix} 21 \\ -7 \end{pmatrix}=\begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Therefore $x=3$, $y=-1$.

Summary

The equations

$$\begin{aligned} ax+by &= p \\ cx+dy &= q \end{aligned}$$

can be represented in matrix form by $\mathbf{A}\mathbf{x}=\mathbf{b}$ where

$$\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{b}=\begin{pmatrix} p \\ q \end{pmatrix}, \quad \mathbf{x}=\begin{pmatrix} x \\ y \end{pmatrix}$$

The solutions $\mathbf{x}$ are obtained from:

$$\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}$$

where the inverse matrix is given by

$$\mathbf{A}^{-1}=\frac{1}{\det\mathbf{A}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}=\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Test yourself!

1. Given that $\mathbf{A} = \begin{bmatrix} 12 & 9 \\ 2 & 3 \end{bmatrix}$, find $\det \mathbf{A}$.
2. Is the matrix $\mathbf{B} = \begin{bmatrix} -25 & 5 \\ 10 & -2 \end{bmatrix}$ singular ?
3. Given that $\mathbf{P} = \begin{bmatrix} 2 & 12 \\ 1 & -6 \end{bmatrix}$, find $\mathbf{P}^{-1}$.
4. Use matrix methods to solve

$$\begin{aligned} 2x + 7y &= -7 \\ 3x - 2y &= 27 \end{aligned}$$

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1. $\det \mathbf{A} = 18$
2. Yes! Because $\det \mathbf{B} = 0$.
3. $\mathbf{P}^{-1} = -\frac{1}{24} \begin{bmatrix} -6 & -12 \\ -1 & 2 \end{bmatrix}$
4. $x = 7$, $y = -3$.