Introduction to Matrices

Algebra 6

INU0114/514 (Maths 1)

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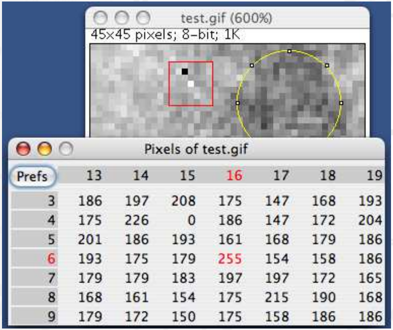
Introduction

Matrices can be used to represent complex data sets such as signals or the pixel values of images.



Matrices are also used to simplify calculations in many areas of science and engineering such as cryptography, quantum mechanics, general relativity, computer graphics, neural networks.

On this course we will see that matrices are a useful way of representing and solving *linear systems* of equations.

Objectives

This presentation will introduce some basic concepts in matrix algebra.

- Element addresses and other conventions
- Matrix arithmetic (addition / subtraction).
- Multiplying a matrix by a constant (scalar product)
- Multiplying matrices together (matrix product)
- The identity matrix

Matrices

Definition of a matrix

A matrix (plural *matrices*) is a set of real or complex numbers (called *elements*) arranged in rows and columns to form a rectangular array.

A matrix having m rows and n columns is referred to as having order (or dimensions) of $m \times n$. A matrix is indicated by writing the array within brackets or paretheses.

For example, here is a 3×4 matrix (3 rows and 4 columns):

$$\begin{bmatrix} 2 & 5 & 1 & -2 \\ 1 & 5 & 3 & 0 \\ 9 & 3 & -6 & 6 \end{bmatrix}$$

Here is a 2×1 matrix (2 rows, 1 column):

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

Matrix notation

Each element within a matrix has its own particular 'address'. Double suffix notation is used to label the elements.

For a general $m \times n$ matrix (m rows and n columns) the addresses are arranged like this:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

So if we have the following matrix

$$\begin{pmatrix} 10 & 8 & 4 & 1 \\ 0 & -3 & 2 & 6 \\ 1 & 3 & -9 & -1 \end{pmatrix}$$

We can refer to specific elements by their address. For example:

$$a_{12} = 8 \qquad a_{21} = 0 \qquad a_{32} = 3 \qquad a_{34} = -1$$

Labelling a matrix

When matrices are involved in arithmetic (as we shall see in the next section) we can use a letter to represent the whole matrix. The convention in books and other printed material is to use a **bold typeface**.

In handwritten work matrices are represented using an underline.

So the previous matrix could be expressed as

$$\mathbf{A} = \begin{pmatrix} 10 & 8 & 4 & 1 \\ 0 & -3 & 2 & 6 \\ 1 & 3 & -9 & -1 \end{pmatrix} \quad \text{in printed documents, or}$$

$$\underline{\mathscr{A}} = \begin{pmatrix} 10 & 8 & 4 & 1 \\ 0 & -3 & 2 & 6 \\ 1 & 3 & -9 & -1 \end{pmatrix} \quad \text{in handwritten documents.}$$

Important!

It is crucial to follow this convention because matrices and non-matrix quantities often appear within equations and must be treated differently.

Adding matrices

Definition

If $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are $m \times n$ matrices, their sum $\mathbf{A} + \mathbf{B}$, is the $m \times n$ matrix obtained by adding corresponding entries:

$$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

For example:

$$\mathbf{A} = \begin{pmatrix} 2 & 5 \\ -1 & 0 \\ 3 & -2 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -5 & 2 \\ 0 & -9 \\ 8 & -1 \end{pmatrix}$$

Add the corresponding elements of each matrix:

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2-5 & 5+2 \\ -1+0 & 0-9 \\ 3+8 & -2-1 \end{pmatrix} = \begin{pmatrix} -3 & 7 \\ -1 & -9 \\ 11 & -3 \end{pmatrix}$$

Matrices can be added (or subtracted) only if the matrices have the same size.

Scalar multiplication

Definition

If k is a constant (often called a *scalar*) and $\mathbf{A} = [a_{ij}]$ is a $m \times n$ matrix, then scalar multiplication $k\mathbf{A}$ is the $m \times n$ matrix obtained by multiplying each element by k:

$$k[a_{ij}] = [ka_{ij}]$$

For example, given the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 4 & -1 \end{pmatrix}$$

Then $5\mathbf{A}$ is the result of multiplying each element in the matrix by 5

$$5\mathbf{A} = 5\begin{pmatrix} 1 & 2 & 3 \\ -2 & 4 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 10 & 15 \\ -10 & 20 & -5 \end{pmatrix}$$

Test yourself

The matrices **A**, **B** and **C** are defined by:

$$\mathbf{A} = \begin{bmatrix} 8 & 0 \\ -2 & 3 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} -2 & 9 \\ 4 & 5 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 1 & 4 \\ -2 & 3 \\ 9 & 0 \end{bmatrix}$$

Calculate the following:

1. $\mathbf{A} + \mathbf{B}$
2. $\mathbf{B} - \mathbf{A}$
3. $\mathbf{B} + \mathbf{C}$
4. $-2\mathbf{A}$

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1. $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 6 & 9 \\ 2 & 8 \end{bmatrix}$
2. $\mathbf{B} - \mathbf{A} = \begin{bmatrix} -10 & 9 \\ 6 & 2 \end{bmatrix}$
3. $\mathbf{B} + \mathbf{C}$ is not defined
4. $-2\mathbf{A} = \begin{bmatrix} -16 & 0 \\ 4 & -6 \end{bmatrix}$

Matrix multiplication

Definition

If $\mathbf{A}$ is a $m \times n$ matrix and $\mathbf{B}$ is a $n \times p$ matrix then

$$\mathbf{C} = \mathbf{AB}$$

is a $m \times p$ matrix.

The elements in $\mathbf{C}$ are computed using

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{in} b_{nj}$$

This means: multiply each element in the i^{th} row of $\mathbf{A}$ with the corresponding element in the j^{th} column of $\mathbf{B}$ and then add the products.

In picture form, here is how rows and columns are combined to form the elements of the new matrix:

$$\mathbf{C} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1p} \\ b_{21} & \dots & b_{2j} & \dots & b_{2p} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \dots & b_{nj} & \dots & b_{np} \end{bmatrix}$$

Multiplication is possible only when the number of rows in **A** is equal to the number of columns in **B**.

Matrix multiplication

Given that $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 5 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 1 \\ 4 & -3 \end{pmatrix}$ find **AB** and **BA**.

The multiplication takes place as follows:

$$\mathbf{AB} = \begin{pmatrix} 1 & -2 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} (1\times 5)+(-2\times 4) & \star \\ \star & \star \end{pmatrix} = \begin{pmatrix} -3 & \star \\ \star & \star \end{pmatrix}$$

$$\mathbf{AB} = \begin{pmatrix} 1 & -2 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} -3 & \star \\ (5\times 5)+(6\times 4) & \star \end{pmatrix} = \begin{pmatrix} -3 & \star \\ 49 & \star \end{pmatrix}$$

$$\mathbf{AB} = \begin{pmatrix} 1 & -2 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} -3 & (1\times 1)+(-2\times -3) \\ 49 & \star \end{pmatrix} = \begin{pmatrix} -3 & 7 \\ 49 & \star \end{pmatrix}$$

$$\mathbf{AB} = \begin{pmatrix} 1 & -2 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} -3 & 7 \\ 49 & (5\times 1)+(6\times -3) \end{pmatrix} = \begin{pmatrix} -3 & 7 \\ 49 & -13 \end{pmatrix}$$

Therefore $\mathbf{AB} = \begin{pmatrix} -3 & 7 \\ 49 & -13 \end{pmatrix}$.

Now let's find **BA**.

$$\mathbf{BA} = \begin{pmatrix} 5 & 1 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 5 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} (5)(1)+(1)(5) & (5)(-2)+(1)(6) \\ (4)(1)+(-3)(5) & (4)(-2)+(-3)(6) \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} 10 & -4 \\ -11 & -26 \end{pmatrix}$$

Since $\mathbf{AB} = \begin{pmatrix} -3 & 7 \\ 49 & -13 \end{pmatrix}$ then we see that $\mathbf{AB} \neq \mathbf{BA}$.

Matrix multiplication is not commutative. In general $\mathbf{AB} \neq \mathbf{BA}$. Matrix algebra has other important differences with the rules of "normal" algebra too!

Two matrices can be multiplied together only if the number of columns in the first matrix is the same as the number of rows in the second matrix.

A quick way to see if matrix multiplication is possible is to write the dimensions of each matrix underneath. For example, given the matrices

$$\mathbf{A} = \begin{pmatrix} \star & \star & \star \\ \star & \star & \star \end{pmatrix} \mathbf{B} = \begin{pmatrix} \star & \star \\ \star & \star \\ \star & \star \end{pmatrix} \mathbf{C} = \begin{pmatrix} \star & \star & \star & \star & \star \end{pmatrix}$$

Suppose we want to calculate the product **AB**. The dimensions are

$$(2 \times \underline{3}) \text{ and } (\underline{3} \times 2)$$

The "inner" pair of numbers in these brackets (underlined) is $(3,3)$. They are the same so we *can* multiply the matrices together.

The "outer" pair $(2,2)$ give the size of the resulting matrix - (2×2).

Consider the product **BC**. The dimensions of these matrices are

$$(3 \times \underline{2}) \text{ and } (\underline{1} \times 5)$$

The "inner" pair of numbers are $(2,1)$. They are different so we *cannot* multiply the matrices together.

Test yourself!

Given the matrices $\mathbf{A} = \begin{bmatrix} 5 & -2 \\ 6 & 1 \\ -5 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 & 4 \\ 8 & -2 & 7 \end{bmatrix}$.

1. What will be the sizes of the matrices **AB** and **BA**?
2. Calculate the matrices **AB** and **BA**.

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1. **AB** is a 3×3 matrix. **BA** is a 2×2 matrix.
2. $\mathbf{AB} = \begin{bmatrix} -11 & 4 & 6 \\ 14 & -2 & 31 \\ 19 & -6 & 1 \end{bmatrix}$, $\mathbf{BA} = \begin{bmatrix} -15 & 10 \\ -7 & 3 \end{bmatrix}$

The identity matrix

Given the matrices $\mathbf{A} = \begin{pmatrix} 10 & 8 & 4 \\ 0 & -3 & 2 \\ 1 & 3 & -9 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

Let's calculate $\mathbf{AI}$ and $\mathbf{IA}$.

$$\mathbf{AI} = \begin{pmatrix} 10 & 8 & 4 \\ 0 & -3 & 2 \\ 1 & 3 & -9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 8 & 4 \\ 0 & -3 & 2 \\ 1 & 3 & -9 \end{pmatrix}$$

$$\mathbf{IA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 10 & 8 & 4 \\ 0 & -3 & 2 \\ 1 & 3 & -9 \end{pmatrix} = \begin{pmatrix} 10 & 8 & 4 \\ 0 & -3 & 2 \\ 1 & 3 & -9 \end{pmatrix}$$

We found that $\mathbf{AI} = \mathbf{A}$ and $\mathbf{IA} = \mathbf{A}$.

The matrix $\mathbf{I}$ is called the identity matrix and it behaves like the number 1 in the number system.

It is is a diagonal matrix where the diagonal elements are all 1. Identity matrices are always represented by $\mathbf{I}$.

The inverse matrix

If two matrices $\mathbf{A}$ and $\mathbf{B}$ can be multiplied together to give the identity matrix $\mathbf{I}$:

$$\mathbf{AB} = \mathbf{I}$$

then $\mathbf{A}$ is the inverse matrix of $\mathbf{B}$ (and vice-versa).

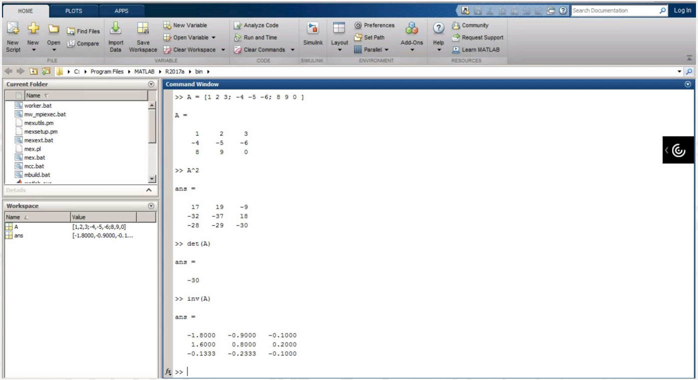
The inverse of a matrix $\mathbf{A}$ is usually denoted by $\mathbf{A}^{-1}$.

E.g. $$\mathbf{A} = \begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix}, \mathbf{A}^{-1} = \begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$$

In this case $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. We will learn how to calculate inverse matrices soon!

Matlab

Matrix calculations can be done in the commercial software package Matlab.



Access via Remote Application Service: `ras.ncl.ac.uk`.

Test yourself

Consider the matrices $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$.

Evaluate the following:

1. $2\mathbf{A} + \mathbf{B}$
2. $\mathbf{B}^2$
3. $\mathbf{AC}$
4. $\mathbf{AB}$

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1. $2\mathbf{A} + \mathbf{B} = \begin{bmatrix} 6 & 3 \\ 1 & 9 \end{bmatrix}$
2. $\mathbf{B}^2 = \mathbf{B} \times \mathbf{B} = \begin{bmatrix} 19 & -15 \\ -5 & 4 \end{bmatrix}$
3. $\mathbf{AC} = \begin{bmatrix} 35 \\ 45 \end{bmatrix}$
4. $\mathbf{AB} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

In (4) we have $\mathbf{AB} = \mathbf{I}$. That means $\mathbf{B}$ is the inverse matrix of $\mathbf{A}$ (and vice-versa).