Inequalities and intervals

Algebra 1

INU0114/514 (Maths 1)

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Overview

In this presentation we will review interval notation and linear inequalities.

- Understand that inequalites are used to compare the sizes of expressions
- Be able to solve linear inequalities
- Understand how to represent intervals using inequalities.

No doubt you have covered these topics before; take the opportunity to review them again.

Definitions

Inequalities

Inequalities are mathematical statements which compare two expressions. Such statements may be true for a range of values of the variable. Inequalities are sometimes called inequations.

Let's start with the basics.

- The expression $a > b$ states that a is greater than b.
- The expression $a < b$ states that a is less than b.
- The expression $a \neq b$ states a is not equal to b (but does not say which is greatest or least).

In all of these 'strict' inequalities a and b are not equal. We also have inequalities in which a and b may be equal:

- The expression $a \geq b$ states that a is greater than or equal to b.
- The notation $a \leq b$ states that a is less than or equal to b.

Reversing an inequality

Let's spend a little time investigating how inequalities behave. We'll use simple examples.

First, notice how inequalities can be *reversed*.

Given that

$$3.5 < 4$$

An equivalent statement is obviously

$$4 > 3.5$$

We can apply this to variable inequalities too:

$$x \geq 5 \quad \Longleftrightarrow \quad 5 \leq x$$

The arrow ($\Longleftrightarrow$) means each inequality implies the other is true.

In general

$$a > b \quad \Longleftrightarrow \quad b < a$$

Manipulating inequalities

Given that

$$7 > 5$$

If we add 7 to both sides:

$$14 > 12$$

the inequality is still valid. Subtract 2 from both sides:

$$12 > 10$$

Multiply (or divide) by a positive number and the inequality remains valid.

For example multiply both sides of the inequality by 2:

$$24 > 20$$

Divide both sides by 4:

$$6 > 5$$

Multipliying or dividing by a negative number invalidates the inequality. For example, multiplying by −3:

$$-18 > -15$$

which is *not* true. So we must reverse the inequality symbol to make it true:

$$-18 < -15$$

Dividing both sides of the previous inequality by −2, and then reversing the inequality we get another valid statement.

$$9 > 7.5$$

Solving linear inequalities

Inequalities can be rearranged and solved in much the same way as equations can be. All we have to do is isolate x. We just have to remember the differences which occur when multiplying or dividing by a negative number.

Solving an inequality

Solve $6-5x \leq 36$.

Subtract 6 from both sides:

$$-5x \leq 30$$

Divide both sides by -5 and *reverse the inequality sign*:

$$x \geq -6$$

Solving an inequality

Solve $20-5x>36-x$.

Add $5x$ to both sides and subtract 36 from both:

$$\begin{aligned} 20-36 &> -x+5x \\ -16 &> 4x \\ -4 &> x \end{aligned}$$

Or reverse the inequality and write:

$$x<-4$$

Compound inequalities

If we want to specify a range of numbers, we can use inequalities to do so. For example, consider the interval defined by the following two inequalities:

$$x \geq -3 \qquad \text{and} \qquad x \leq 2$$

Taken together, these two inequalities specify an interval containing all the numbers between −3 and 2 inclusive.

If we reverse the first inequality sign we can write the previous pair of inequalities as:

$$-3 \leq x \qquad \text{and} \qquad x \leq 2 \tag{1}$$

And then combine them into a single mathematical statement:

$$-3 \leq x \leq 2$$

This is a *compound inequality* and it is equivalent to the two inequalities given in equation 1. The compound inequality is read as "−2 is less than or equal to x, which is less than or equal to 3".

Solving compound inequalities

Solving a compound inequality

Find the values of x for which

$$0 < 5(x+3) \leq 10$$

To solve: split the compound inequality into two simple inequalities

$$0 < 5(x+3) \quad \text{and} \quad 5(x+3) \leq 10$$

Solve each inequality to get:

$$x > -3 \quad \text{and} \quad x \leq -1$$

We can combine these two statements into another compound inequality to get the solution:

$$-3 < x \leq -1$$

Interval notation

Frequently in maths we need to specify a range of values. Some mathematical functions might only accept certain values as their input, or we might want to restrict the answers to a problem to a certain range of values. Mathematicians use *interval notation* to do this.

Square brackets [or] denote end points which are included in the interval. On a number line shaded circles are used to denote them.

Round brackets (or) denote end points which are not included in the interval. On a number line unshaded circles are used to denote them.

Consider the compound inequality

$$4 \leq x < 9$$

The range of values described here goes from exactly 4 all the way up to, but not including 9. So 8.999999 is included but 9 is not. With interval notation this is written as

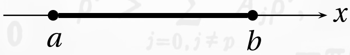
$$[4, 9)$$

Open and closed intervals

An open interval does not include the end points.

The open interval is represented by (a, b) or by the inequality $a < x < b$.

A closed interval includes the end points.



The closed interval is represented by $[a, b]$ or by the inequality $a \leq x \leq b$.

Semi-open intervals

There are two versions of this interval. In the first the interval includes b but not a.

This semi-open interval is represented by $(a, b]$ or by the inequality $a < x \leq b$.

In the second version, the open and closed endpoints are simply reversed.

This interval includes a but not b.

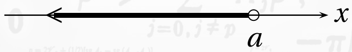
This semi-open interval is represented by $[a, b)$ or by the inequality $a \leq x < b$.

Non-ending intervals

This interval includes a. Infinity cannot be included.

This interval is represented by $[a, \infty)$ or by the inequality $x \geq a$.

Another non-ending interval is shown below. This interval excludes a. Infinity (minus) cannot be included.



This interval is represented by $(-\infty, a)$ or by the inequality $x < a$.

Combining intervals

Sometimes it is necessary to express combinations of intervals.

For example, how can inequalities and interval notation be used to express the statement "All values except 5?"

With inequalities, we could write

$$x < 5 \text{ and } x > 5$$

The equivalent interval notation is written like this:

$$(-\infty, 5) \cup (5, \infty)$$

The word "and" has been replaced with a symbol $\cup$ which means the same and stands for *union*.

Another alternative:

$$x \neq 5$$

Test yourself...

You should be able to solve the following problems if you have understood everything in these notes.

1. Solve $5x+3>23$.
2. Solve $2-4x\geq 10$.
3. Solve $-14y-6\leq 1-3y$
4. Express

   as a compound inequality
5. Solve $-4<2x+1<9$

Answers:

1. $x>4$
2. $x\leq -2$
3. $y\geq -\frac{7}{11}$
4. $-2<x\leq 3$
5. $-\frac{5}{2}<x<4$