Basic algebra

Algebra 1

INU0114/514 (Maths 1)

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Overview

In this presentation we will review some basic definitions and skills required to do algebra.

- Expanding brackets and factorising expressions
- Solving equations
- Identities
- Using and rearranging formulae
- Algebraic fractions

No doubt you have covered these topics before; take the opportunity to review them again.

Expressions

Expression

An expression is simply a combination of mathematical symbols, usually consisting of operators (+, −, ×, ÷) and letters representing unknown variables. Expressions can also contain more complex functions (such as trigonometric functions).

For example:

$$5x^2 + 2xy - \cos x$$

This expression consists of 3 *terms* separated by a "+" and a "−" sign. In the second term: $2xy$ actually represents a multiplication $2 \times x \times y$.

The power only refers to the number or variable it is written above. In the first term of the expression above $5x^2$ means $5 \times x \times x$.

Simplifying expressions

Since the order doesn't matter in multiplication we can simplify terms by rearranging their order. For example

$$\begin{aligned} 8p \times 3q \quad \text{means} \quad & 8 \times p \times 3 \times q \\ = \quad & 8 \times 3 \times p \times q \\ = \quad & 24pq \end{aligned}$$

The terms in an expression can be added or subtracted if they are *like terms*. Like terms contain the same combination of letters. For example $3ab$ and $5ab$ are like terms and can be added or subtracted:

$$3ab + 5ab = 8ab$$

Or

$$9x^2 - 7x^2 = 2x^2$$

Unlike terms cannot be combined or simplified any further. For example, we cannot simplify $3x + y$ or $2x + 3x^2$.

Factorising expressions

Expressions can be factorised if some or all of the terms have a common factor.

Factorising expressions

Factorise $3x+6$.

Both terms are divisible by 3, so taking a common factor of 3 we can write

$$3x+6 \equiv 3(x+2)$$

Note the use of the identity symbol ($\equiv$) — we use it because expressions are not equations. More about this shortly.

Factorising expressions

Factorise $2x^3+8x^2-6x$.

All terms are divisible by 2 and x, so taking a common factor of $2x$ we can write

$$2x^3+8x^2-6x \equiv 2x(x^2+4x-3)$$

Equations

Equation

An equation is a statement that two mathematical expressions are equal to each other.

$$3x+10=x+25 \quad \cos x=\tfrac{1}{2} \quad \log x=3$$

The statement not true for all values of the variable. (x, in this case)

We follow a procedure (called algebra) to discover the value of the unknown x in the equation. More about this later.

Solving equations

The rules

Equations are solved by rearranging to make the unknown quantity the subject of the formula. When solving equations you must make sure that the statements on each side stay equal to each other. To do this you should follow these rules:

- Add the same thing to both sides.
- Subtract the same thing from both sides.
- Multiply both sides by the same amount.
- Divide both sides by the same amount (but don't divide by zero).
- Square both sides, square-root both sides, etc.

In other words, to maintain consistency we must carry out the same operation on both sides of an equation.

Solving a simple equation

Solve the equation $5x - 4 = 3x + 12$

Try to gather the unknowns (the x's) on one side of the equation and the *knowns* on the other.

Subtract $3x$ from both sides:

$$\begin{aligned} 5x - 4 - 3x &= 3x + 12 - 3x \\ 2x - 4 &= 12 \end{aligned}$$

Add 4 to both sides:

$$\begin{aligned} 2x - 4 + 4 &= 12 + 4 \\ 2x &= 16 \end{aligned}$$

Divide both sides by 2

$$\begin{aligned} \frac{2x}{2} &= \frac{16}{2} \\ x &= 8 \end{aligned}$$

Verifying a solution

In the previous example we solved the equation

$$5x - 4 = 3x + 12$$

and found that $x = 8$ was the solution.

How do we know it is the solution to the equation?

We check it by substituting back into the equation:

$$\begin{aligned} (5 \times 8) - 4 &= (3 \times 8) + 12 \\ 36 &= 36 \end{aligned}$$

Both sides of the equation are the same, so $x = 8$ is definitely a solution!

Solving equations quickly

We didn't need to write down so many steps in the previous example; some of them can be safely carried out mentally.

You can see that subtracting (or adding) the same amount to both sides has the same effect as shifting the terms from one side of the equation to the other.

This gives an alternative way to isolate the subject of an equation; we can *rearrange* the terms.

A more condensed form of the solution to the same problem is this:

$$5x - 4 = 3x + 12$$

Move the $3x$ from the RHS to the LHS, where it will be subtracted:

$$2x - 4 = 12$$

Move the -4 to the RHS where it will add to that side:

$$2x = 16$$

Divide the 2 onto the RHS to get the solution $x = 8$.

It takes time and practice to become confident at solving equations and it is likely that you have already had much experience already!

Identities

Identity

An identity is a statement that two mathematical expressions are equal for all values of the variable

$$(x+1)^2 \equiv x^2+2x+1 \quad \tan x \equiv \frac{\sin x}{\cos x} \quad a^2-b^2 \equiv (a+b)(a-b)$$

(*a*, *b* and *x*, in this case). Identities are indicated by the $\equiv$ symbol.

Identities are different to equations — we can't solve an identity. It is just a statement which is true for every possible value of the the variable.

Sometimes mathematicians are lazy and use the equal sign (=) when they really mean the identity ($\equiv$).

It is important that you understand the difference.

Formulae

Formulae

A formula is way of expressing a general relationship between quantities.

$$E = mc^2 \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad v^2 = u^2 + 2as$$

For example, substituting values of m and c into the first formula, we could calculate the value of E.

A formula is a type of equation. Formulas are useful to scientists because they allow quantities to be calculated from other known quantities.

For example, if we know the mass m of a particle and the speed of light c then we can calculate the energy E associated with it.

Formulae

Using a formula

A conversion formula

Here is a temperature formula

$$F = \frac{9}{5}C + 32$$

for converting between units of Celsius to Fahrenheit.

For example, a temperature of 20°C:

$$F = \frac{9}{5} \times 20 + 32 = 36 + 32 = 68$$

Therefore 20°C is equivalent to 68°F.

Calculating centripetal force

An object moving in a circle is experiencing a force towards the centre of the circle, called centripetal force, given by

$$F = \frac{mv^2}{R}$$

where m is the mass of the object, v is the object speed and R is the circle radius. Find F when $m = 10$, $v = 5$ and $R = 50$.

Substitute the values into the RHS of the formula:

$$F = \frac{10 \times 5^2}{50} = \frac{250}{50} = 5$$

Rearranging a formula

Given the formula $T = 2\pi\sqrt{\frac{L}{g}}$ rearrange it to make L the subject.

Divide both sides by 2π:

$$\frac{T}{2\pi} = \sqrt{\frac{L}{g}}$$

Square both sides to undo the square-root:

$$\left(\frac{T}{2\pi}\right)^2 = \left(\sqrt{\frac{L}{g}}\right)^2 \quad \Rightarrow \frac{T^2}{4\pi^2} = \frac{L}{g}$$

Finally, cross multiply by g to get

$$L = \frac{gT^2}{4\pi^2}$$

Algebraic fractions

We frequently have to combine and simplify algebraic fractions. We do this by putting them over a common denominator.

Example

Simplify $\frac{x}{2x+1} + \frac{3}{x-1}$.

Put the fractions over a common denominator:

$$\begin{aligned} \frac{x}{2x+1} + \frac{3}{x-1} &\equiv \frac{x(x-1)+3(2x+1)}{(2x+1)(x-1)} \\ &\equiv \frac{x^2-x+6x+3}{(2x+1)(x-1)} \\ &\equiv \frac{x^2+5x+3}{(2x+1)(x-1)} \end{aligned}$$

The process is exactly the same as that for adding (or subtracting) numerical fractions (e.g. $\frac{2}{5}+\frac{1}{3}$).

Test yourself...

You should be able to solve the following problems if you have understood everything in these notes.

1. Simplify $5ab + 2a^2 - (4ab - 3a^2)$.
2. Factorise the answer to (1).
3. Solve $18x + 5 = 10x - 35$.
4. Given that $E = \frac{1}{2}mv^2$, make v the subject of the formula.
5. Is $x = 4$ a solution to $x^2 + 6x - 20 = 0$?
6. Simplify $\dfrac{8}{x-2} - \dfrac{2}{x+1}$.

Answers:

1. $ab + 5a^2$
2. $a(b + 5a)$
3. $x = -5$
4. $v = \sqrt{\frac{2E}{m}}$
5. Sub. into LHS and show that $20 = 0$; therefore $x = 4$ is not a solution to the equation.
6. $\dfrac{6(x+2)}{(x-2)(x+1)}$